MECHANICAL VIBRATIONS

Course Name: B.Tech-ME

Semester: 7th

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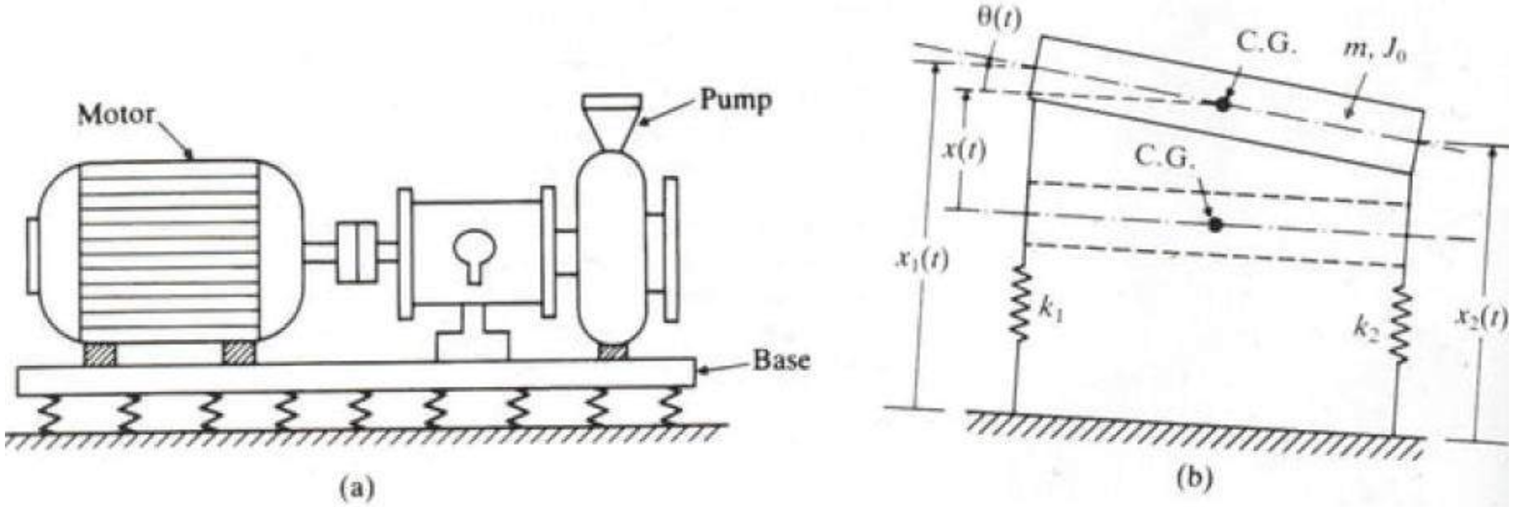
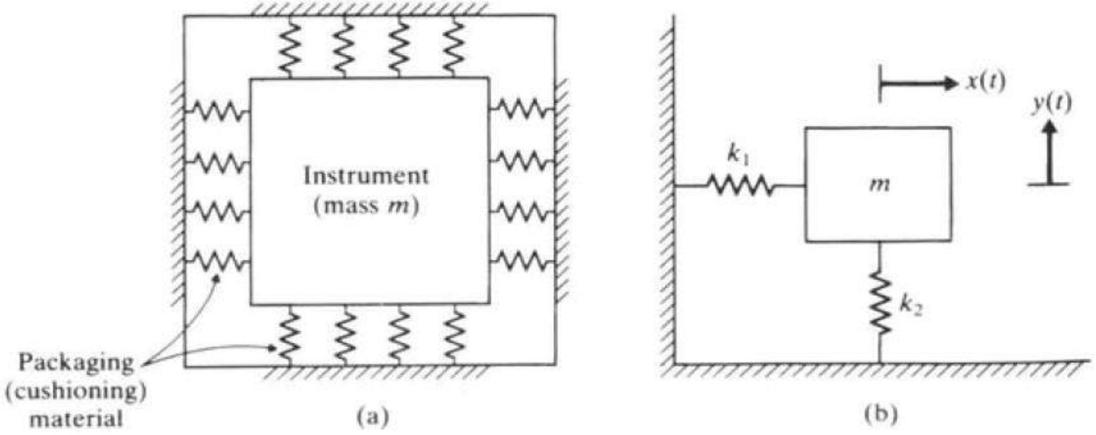
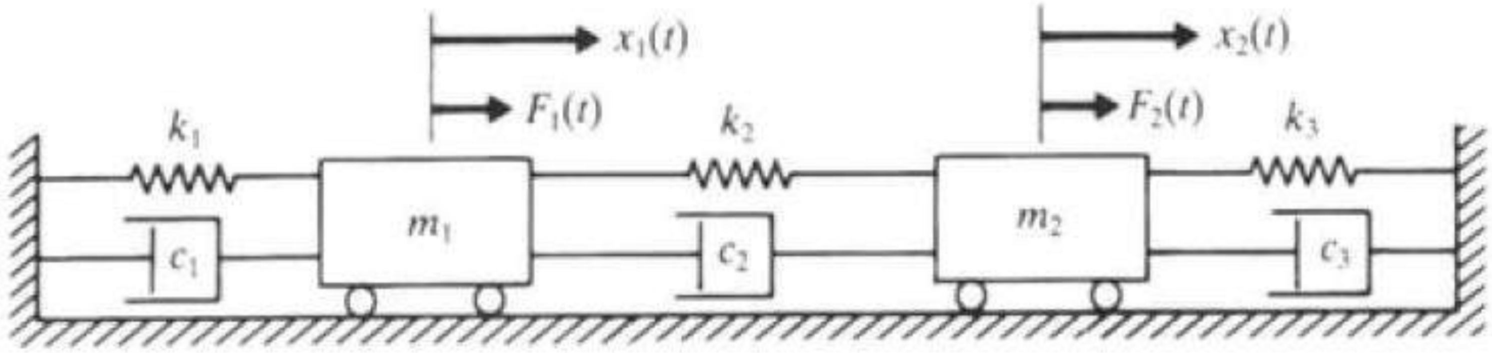
UNIT: III Two Degree of Freedom Systems

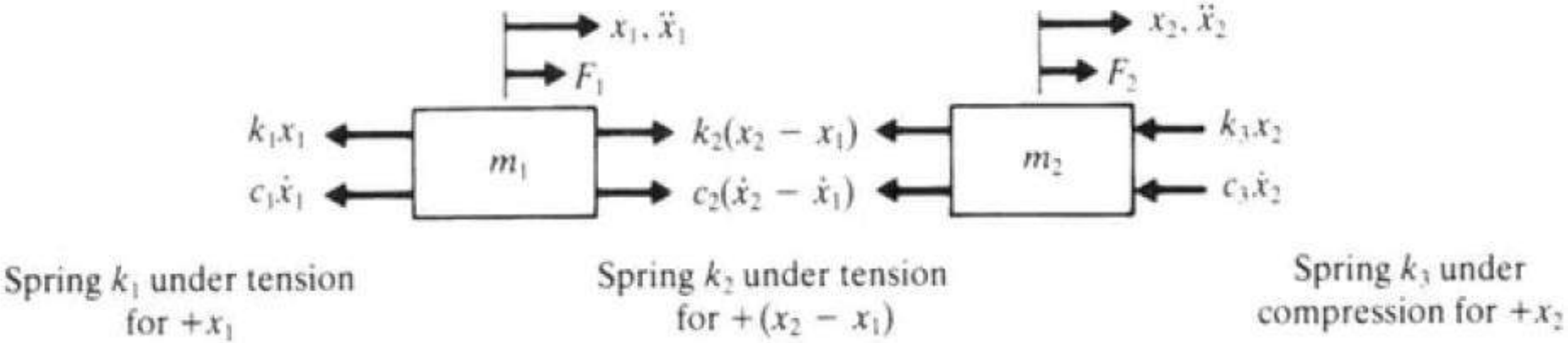




- No. of DoF of system = No. of mass elements x number of motion types for each mass
- For each degree of freedom there exists an equation of motion – usually **coupled** differential equations.
- Coupled means that the motion in one coordinate system depends on the other
- If harmonic solution is assumed, the equations produce two natural frequencies and the amplitudes of the two degrees of freedom are related by the *natural, principal or normal* mode of vibration.
- Under an arbitrary initial disturbance, the system will vibrate freely such that the two normal modes are superimposed.
- Under sustained harmonic excitation, the system will vibrate at the excitation frequency. Resonance occurs if the excitation frequency corresponds to one of the natural frequencies of the system

- **Equations of motion**
- Consider a viscously damped system:
- Motion of system described by position $x_1(t)$ and $x_2(t)$ of masses m_1 and m_2
- The free-body diagram is used to develop the equations of motion using Newton's second law





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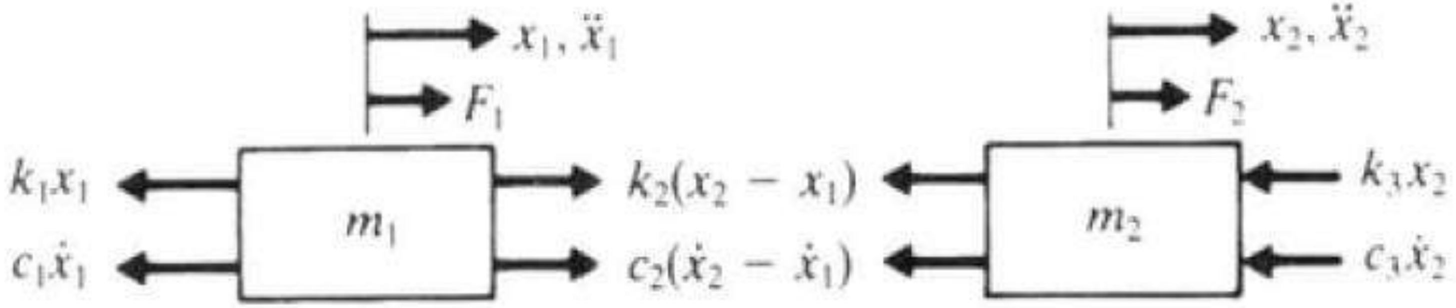
- **Equations of motion**



$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 - c_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) = F_1$$
$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) + c_3\dot{x}_2 + k_3x_2 = F_2$$

or

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1$$
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = F_2$$

- The differential equations of motion for mass m_1 and mass m_2 are **coupled**.
- The motion of each mass is influenced by the motion of the other.

- **Equations of motion**

$$m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1-c_2\dot{x}_2+(k_1+k_2)x_1-k_2x_2=F_1$$
$$m_2\ddot{x}_2-c_2\dot{x}_1+(c_2+c_3)\dot{x}_2-k_2x_1+(k_2+k_3)x_2=F_2$$

- The coupled differential eqns. of motion can be written in matrix form:

$$[m]\ddot{\vec{x}}(t)+[c]\dot{\vec{x}}(t)+[k]\vec{x}(t)=\vec{F}(t)$$

where $[m]$, $[c]$ and $[k]$ are the mass, damping and stiffness matrices respectively and are given by:

$$[m]=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad [c]=\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix} \qquad [k]=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}$$

$\vec{x}(t)$, $\dot{\vec{x}}(t)$, $\ddot{\vec{x}}(t)$ and $\vec{F}(t)$ are the displacement, velocity, acceleration and force vectors respectively and are given by :

$$\vec{x}(t)=\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \quad \dot{\vec{x}}(t)=\begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} \quad \ddot{\vec{x}}(t)=\begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix} \quad and \quad \vec{F}(t)=\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

- Note: the mass, damping and stiffness matrices are all square and symmetric $[m]=[m]^T$ and consist of the mass, damping and stiffness constants.

UNIT: III

Free Vibrations of Undamped System

- The eqns. of motion for a free and undamped TDoF system become:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$
$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = 0$$

- Let us assume that the resulting motion of each mass is harmonic: For simplicity, we will also assume that the response frequencies and phase will be the same:

$$x_1(t) = X_1 \cos(\omega t + \phi) \quad \text{and} \quad x_2(t) = X_2 \cos(\omega t + \phi)$$

- Substituting the assumed solutions into the eqns. of motion:

$$\left[\left\{-m_1\omega^2 + (k_1 + k_2)\right\} X_1 - k_2X_2\right]\cos(\omega t + \phi) = 0$$
$$\left[-k_2X_1 + \left\{-m_2\omega^2 + (k_2 + k_3)\right\} X_2\right]\cos(\omega t + \phi) = 0$$

As these equations must be zero for all values of t, the cosine terms cannot be zero. Therefore:

$$\left\{-m_1\omega^2 + (k_1 + k_2)\right\} X_1 - k_2X_2 = 0$$
$$-k_2X_1 + \left\{-m_2\omega^2 + (k_2 + k_3)\right\} X_2 = 0$$

- Represent two simultaneous algebraic equations with a trivial solution when X_1 and X_2 are both zero – no vibration.

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Free Vibrations of Undamped System

- Written in matrix form it can be seen that the solution exists when the determinant of the mass / stiffness matrix is zero:

$$\begin{bmatrix} \left\{-m_1\omega^2+(k_1+k_2)\right\} & -k_2 \\ -k_2 & \left\{-m_2\omega^2+(k_2+k_2)\right\} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

or

$$m_1m_2\omega^4-\left\{(k_1+k_2)m_2+(k_2+k_3)m_1\right\}\omega^2+(k_1+k_2)(k_2+k_2)-k_2^2=0$$

- The solution to the ***characteristic equation*** yields the natural frequencies of the system.
- The roots of the characteristic equation are:

$$\omega_1^2,\omega_2^2=\frac{1}{2}\left\{\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1m_2}\right\}$$

$$\pm\frac{1}{2}\left[\left\{\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1m_2}\right\}^2-4\left\{\frac{(k_1+k_2)(k_2+k_3)-k_2^2}{m_1m_2}\right\}\right]^{1/2}$$

- This shows that the homogenous solution is harmonic with natural frequencies ω_1 and ω_2

Free Vibrations of Undamped System

- Because the system is coupled, the constants X_1 and X_2 are a function of both natural frequencies ω_1 and ω_2
- Let the values of X_1 and X_2 corresponding to ω_1 be $X_1^{(1)}$ and $X_2^{(1)}$ and *those* corresponding to ω_2 be $X_1^{(2)}$ and $X_2^{(2)}$
- Since the simultaneous algebraic equations are homogeneous only the ***amplitude ratios*** $r_1 = (X_2^{(1)}/X_1^{(1)})$ and $r_2 = (X_2^{(2)}/X_1^{(2)})$ can be determined.
- Substituting ω_1 and ω_2 gives:

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_1^2 + (k_2 + k_3)}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\omega_2^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_2^2 + (k_2 + k_3)}$$

$$\left\{-m_1\omega^2 + (k_1 + k_2)\right\} X_1 - k_2 X_2 = 0$$

$$-k_2 X_1 + \left\{-m_2\omega^2 + (k_2 + k_3)\right\} X_2 = 0$$

- The normal modes of vibration corresponding to the natural frequencies ω_1 and ω_2 can be expressed in vector form known as the ***modal vectors:***

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \\ r_1 X_1^{(1)} \end{Bmatrix} \quad \text{and} \quad \vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \\ r_2 X_1^{(2)} \end{Bmatrix}$$

- The modal vectors describe the ***relative amplitude*** of vibration of each mass for each of the natural frequencies.

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Free Vibrations of Undamped System

- The motion (free vibration) of each mass is given by:

$$\vec{x}^{(1)}(t)=\begin{Bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{Bmatrix}=\begin{Bmatrix} X_1^{(1)}\cos(\omega_1 t+\phi_1) \\ r_1 X_1^{(1)}\cos(\omega_1 t+\phi_1) \end{Bmatrix} \rightarrow First\ mod\ e$$

$$\vec{x}^{(2)}(t)=\begin{Bmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{Bmatrix}=\begin{Bmatrix} X_1^{(2)}\cos(\omega_2 t+\phi_2) \\ r_2 X_1^{(2)}\cos(\omega_2 t+\phi_2) \end{Bmatrix} \rightarrow First\ mod\ e$$

- The constants $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 *and* ϕ_2 are determined from the initial conditions.

Free Vibrations of Undamped System

- Two initial conditions for each mass need to be specified (second order D.E.s)
- The system can be made to vibrate freely in either mode *(i = 1, 2)* by applying the appropriate initial conditions

$$x_1(t=0)=X_1^{(i)} \qquad \dot{x}_1(t=0)=0$$
$$x_2(t=0)=r_i X_1^{(i)} \qquad \dot{x}_2(t=0)=0$$

- Any other combination of initial conditions will result in the excitation of both modes
- Two initial conditions for each mass need to be specified (second order D.E.s)
- The resulting motion is obtained by superposition of the normal modes:

$$\vec{x}(t)=\vec{x}^{(1)}(t)+\vec{x}^{(2)}(t)$$

or

$$\vec{x}_1(t)=\vec{x}_1^{(1)}(t)+\vec{x}_1^{(2)}(t)=X_1^{(1)}\cos(\omega_1 t+\phi_1)+X_1^{(2)}\cos(\omega_2 t+\phi_2)$$
$$\vec{x}_2(t)=\vec{x}_2^{(1)}(t)+\vec{x}_2^{(2)}(t)=r_1 X_1^{(1)}\cos(\omega_1 t+\phi_1)+r_2 X_1^{(2)}\cos(\omega_2 t+\phi_2)$$

- If the initial conditions are:

$$x_1(t=0)=x_1(0) \qquad \dot{x}_1(t=0)=\dot{x}_1(0)$$
$$x_2(t=0)=x_2(0) \qquad \dot{x}_2(t=0)=\dot{x}_2(0)$$

- The constants $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 *and* ϕ_2 can be by substituting the initial conditions in the combined motion eqns.

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Free Vibrations of Undamped System

$$x_1(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$\ddot{x}_2(t) = r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

substituting the initial conditions:

$$x_1(0) = X_1^{(1)} \cos(\phi_1) + X_1^{(2)} \cos(\phi_2)$$

$$\dot{x}_1(0) = -\omega_1 X_1^{(1)} \sin(\phi_1) - \omega_2 X_1^{(2)} \sin(\phi_2)$$

$$x_2(0) = r_1 X_1^{(1)} \cos(\phi_1) + r_2 X_1^{(2)} \cos(\phi_2)$$

$$\dot{x}_2(0) = -\omega_1 r_1 X_1^{(1)} \sin(\phi_1) - \omega_2 r_2 X_1^{(2)} \sin(\phi_2)$$

The following unknowns can be identified:

$$x_1(0) = X_1^{(1)} \cos(\varphi_1) + X_1^{(2)} \cos(\varphi_2)$$

$$\dot{x}_1(0) = -\omega_1 X_1^{(1)} \sin(\varphi_1) - \omega_2 X_1^{(2)} \sin(\varphi_2)$$

$$x_2(0) = r_1 X_1^{(1)} \cos(\varphi_1) + r_2 X_1^{(2)} \cos(\varphi_2)$$

$$\dot{x}_2(0) = -\omega_1 r_1 X_1^{(1)} \sin(\varphi_1) - \omega_2 r_2 X_1^{(2)} \sin(\varphi_2)$$

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Free Vibrations of Undamped System

- Solving for the identified constants yields:

$$X_1^{(1)}\cos(\phi_1)=\left\{\frac{r_2x_1(0)-x_2(0)}{r_2-r_1}\right\} \qquad X_1^{(2)}\cos(\phi_2)=\left\{\frac{-r_1x_1(0)+x_2(0)}{r_2-r_1}\right\}$$

$$X_1^{(1)}\sin(\phi_1)=\left\{\frac{-r_2\dot{x}_1(0)+\dot{x}_2(0)}{\omega_1(r_2-r_1)}\right\} \qquad X_1^{(2)}\sin(\phi_2)=\left\{\frac{r_1\dot{x}_1(0)-\dot{x}_2(0)}{\omega_2(r_2-r_1)}\right\}$$

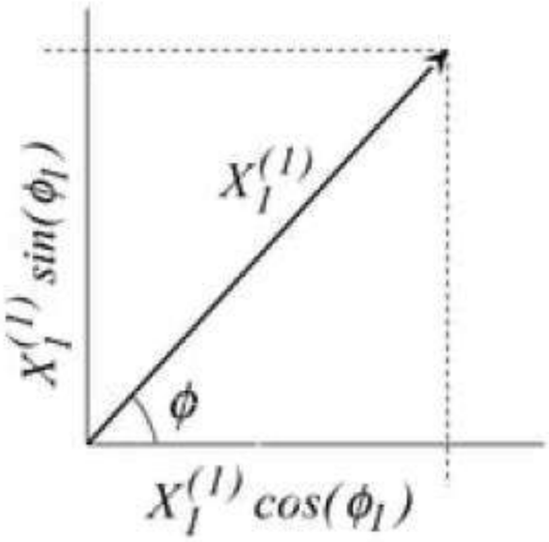
Therefore:

$$X_1^{(1)}=\sqrt{\left\{X_1^{(1)}\cos(\phi_1)\right\}^2+\left\{X_1^{(1)}\sin(\phi_1)\right\}^2}$$

$$X_1^{(2)}=\sqrt{\left\{X_1^{(2)}\cos(\phi_2)\right\}^2+\left\{X_1^{(2)}\sin(\phi_2)\right\}^2}$$

$$\phi_1=a\tan\left\{\frac{X_1^{(1)}\sin(\phi_1)}{X_1^{(1)}\cos(\phi_1)}\right\}$$

$$\phi_2=a\tan\left\{\frac{X_1^{(2)}\sin(\phi_2)}{X_1^{(2)}\cos(\phi_2)}\right\}$$



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Free Vibrations of Undamped System

- In terms of the amplitude ratios r_i and natural frequencies ω_i:

$$X_1^{(1)} = \frac{1}{(r_2 - r_1)}\sqrt{\{r_2 x_1(0) - x_2(0)\}^2 + \frac{\{-r_2 \dot{x}_1(0) + \dot{x}_2(0)\}^2}{\omega_1^2}}$$

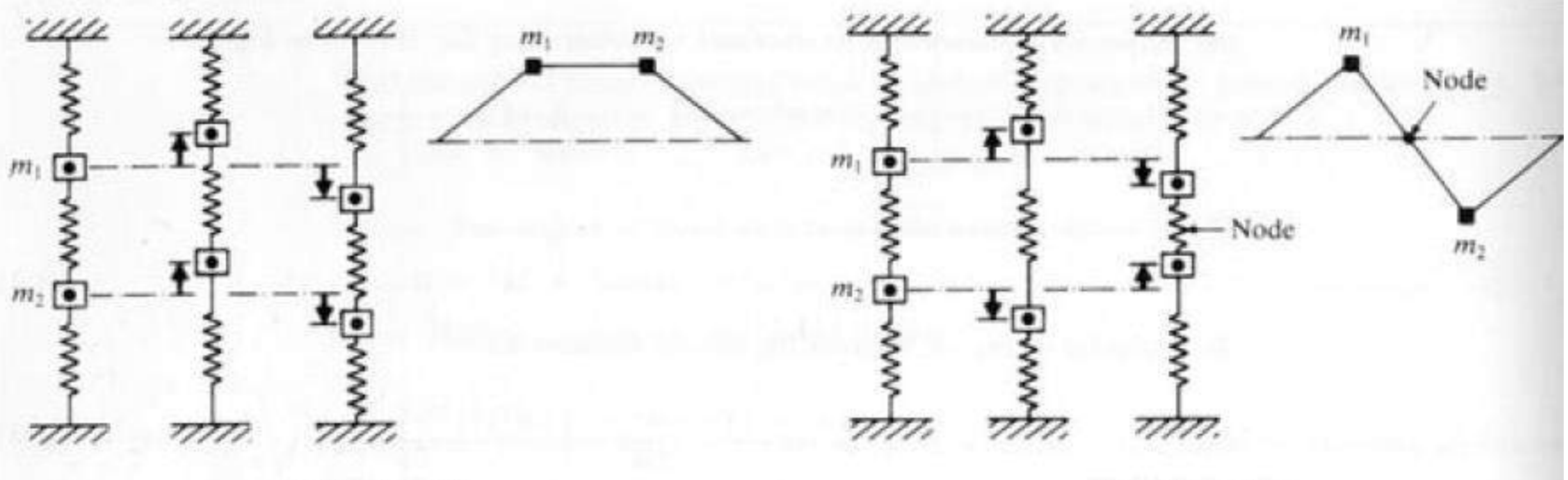
$$X_1^{(2)} = \frac{1}{(r_2 - r_1)}\sqrt{\{-r_1 x_1(0) - x_2(0)\}^2 + \frac{\{r_1 \dot{x}_1(0) + \dot{x}_2(0)\}^2}{\omega_2^2}}$$

$$\phi_1 = a\tan\left\{\frac{-r_2 \dot{x}_1(0) + \dot{x}_2(0)}{\omega_1[r_2 x_1(0) - x_2(0)]}\right\}$$

$$\phi_2 = a\tan\left\{\frac{r_1 \dot{x}_1(0) + \dot{x}_2(0)}{\omega_2[-r_1 x_1(0) - x_2(0)]}\right\}$$

Free Vibrations of Undamped System

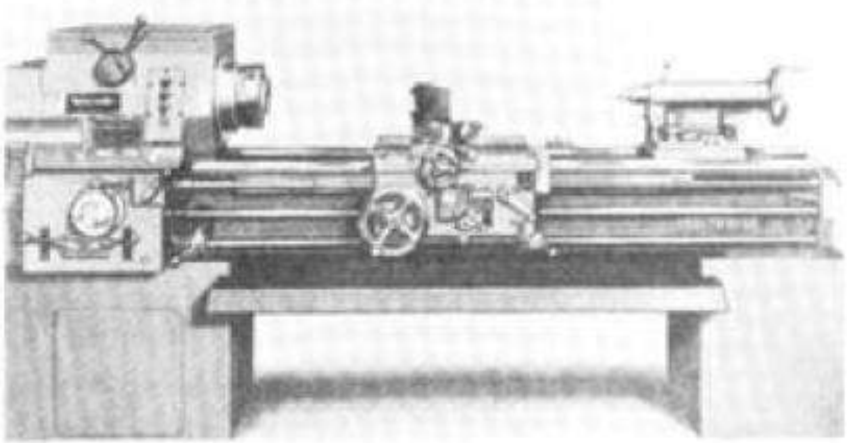
- Example:

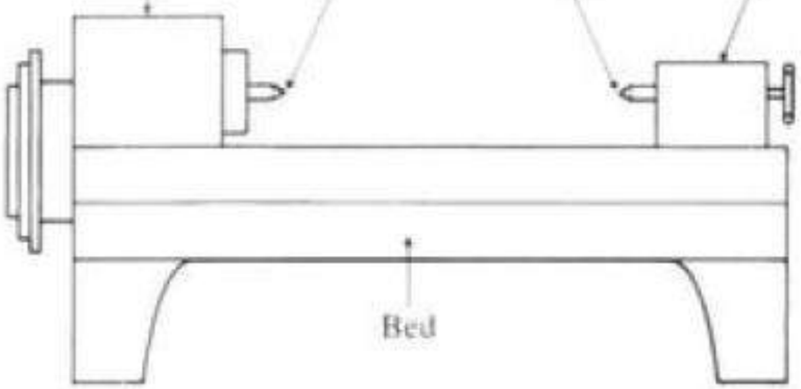


(a) First mode (b) Second mode

Co-ordinate coupling

- Whenever possible, the coordinates are chosen so that they are independent based from the equilibrium position.
- In some cases, another pair of coordinates may be used – ***generalised coordinates***

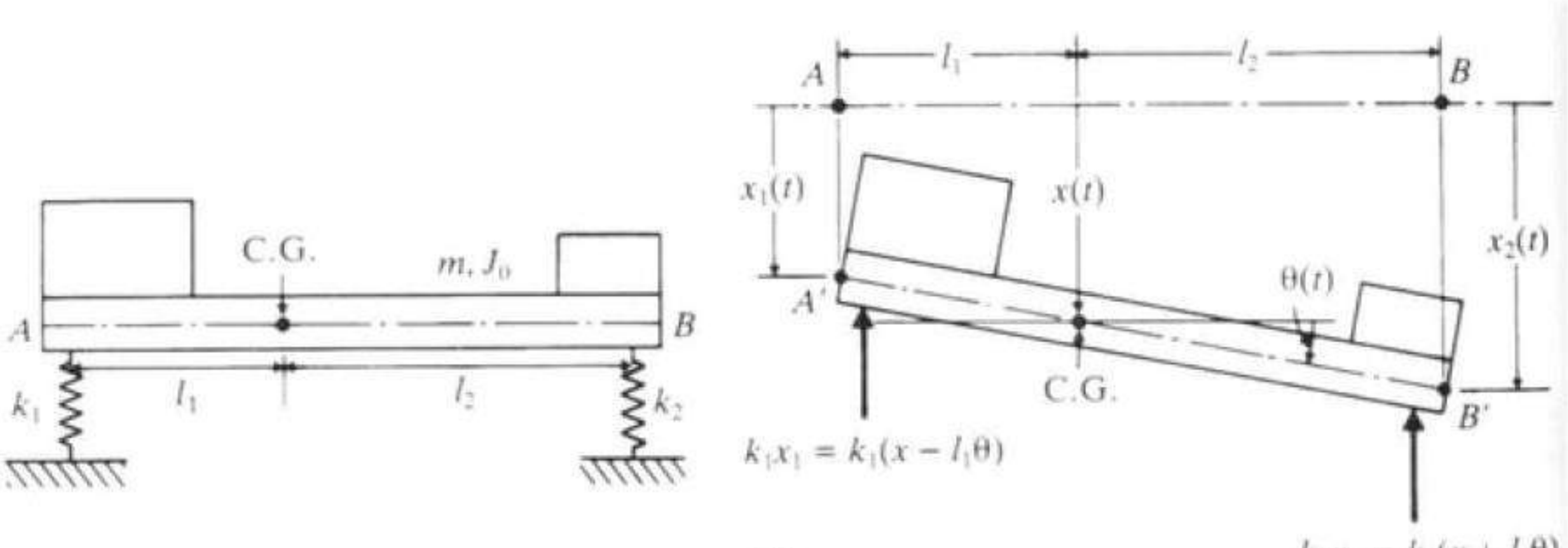




- The lathe can be simplified to be represented by a 2DoF with the bed considered as a rigid body with two lumped masses representing the headstock and tailstock assemblies. The supports are represented by two springs.
- The following set of coordinates can be used to describe the system:

Co-ordinate coupling

- (1): the deflection at each extremity of the lathe $x_1(t)$ and $x_2(t)$
- (2): the deflection at the centre of gravity $x(t)$ and the rotation $\theta(t)$
- (3): the deflection at extremity A $x_1(t)$ and the rotation $\theta(t)$



(a)

UNIT: III

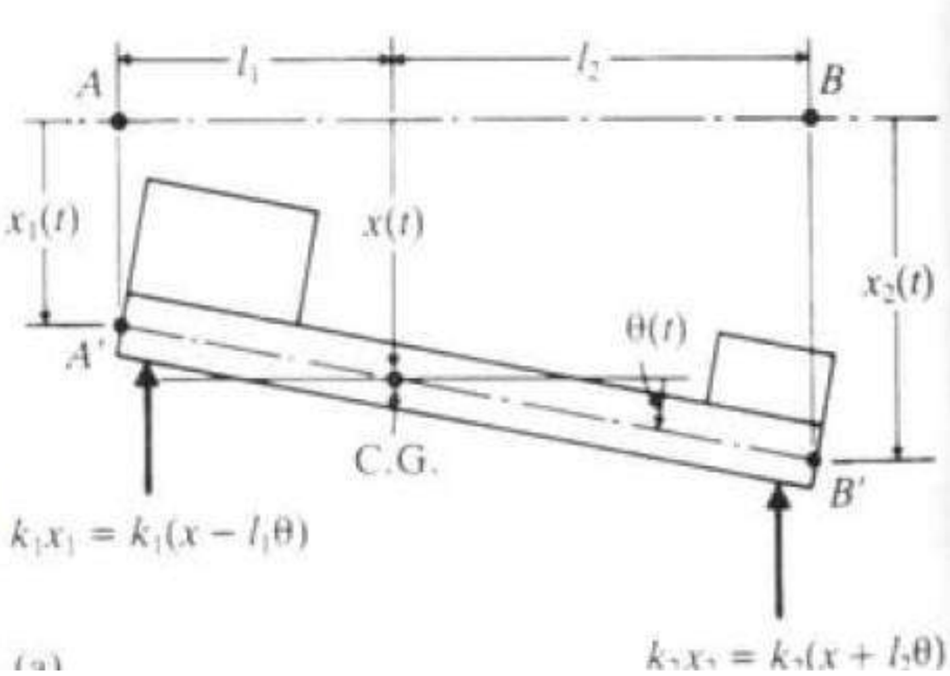
Co-ordinate coupling

- Equations of motion using *x(t)* and *θ(t)*
- Using the FBD, in the vertical direction and about the C.G. respectively:
- $$m\ddot{x} = -k_1(x - l_1\theta) - k_2(x + l_2\theta) \quad and \quad J_o\ddot{\theta} = k_1(x - l_1\theta)l_1 - k_2(x + l_2\theta)l_2$$

in matrix form:

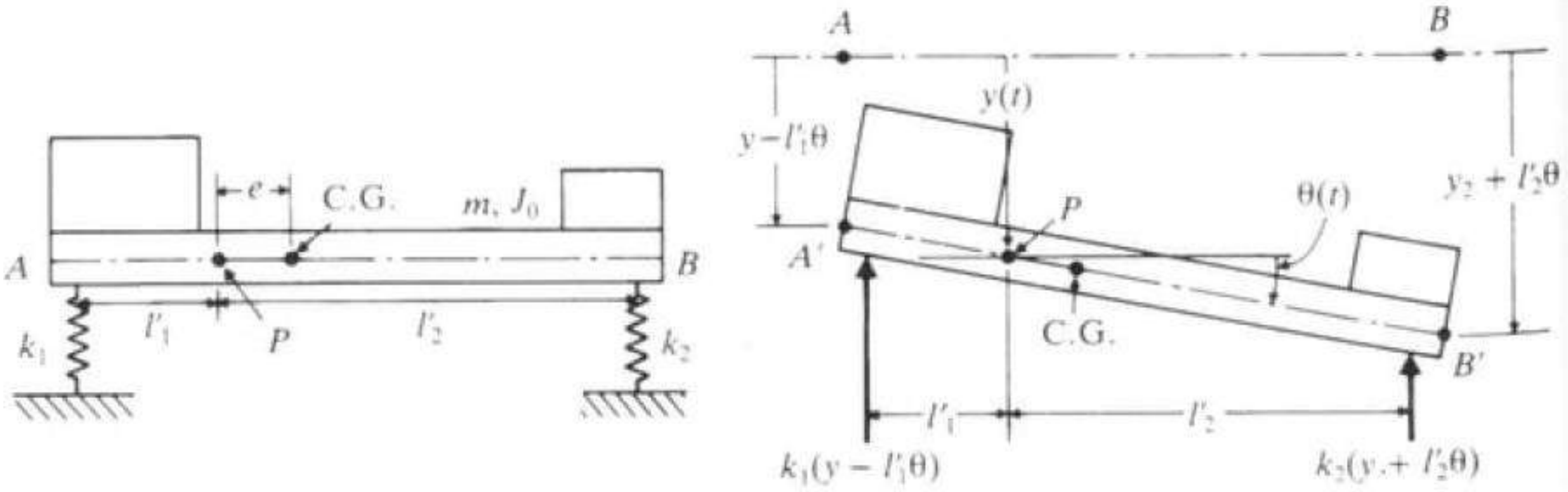
$$\begin{bmatrix} m & 0 \\ 0 & J_o \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- As each eqn. contains both x and θ the system is coupled – ***Elastic*** or ***static coupling***
- Whenever a displacement or torque is applied thru the C.G. the resulting motion will contain **both** translation and rotation.
- The system is uncoupled (eqns. independent) **only** when $k_1 l_1 = k_2 l_2$
- Only then can pure translation or rotation be generated by a displacement or torque thru the C.G.



Co-ordinate coupling

- (1): the deflection *y(t)* at point P located at distance *e* to the left of the C.G. and the rotation *θ(t)*



Co-ordinate coupling

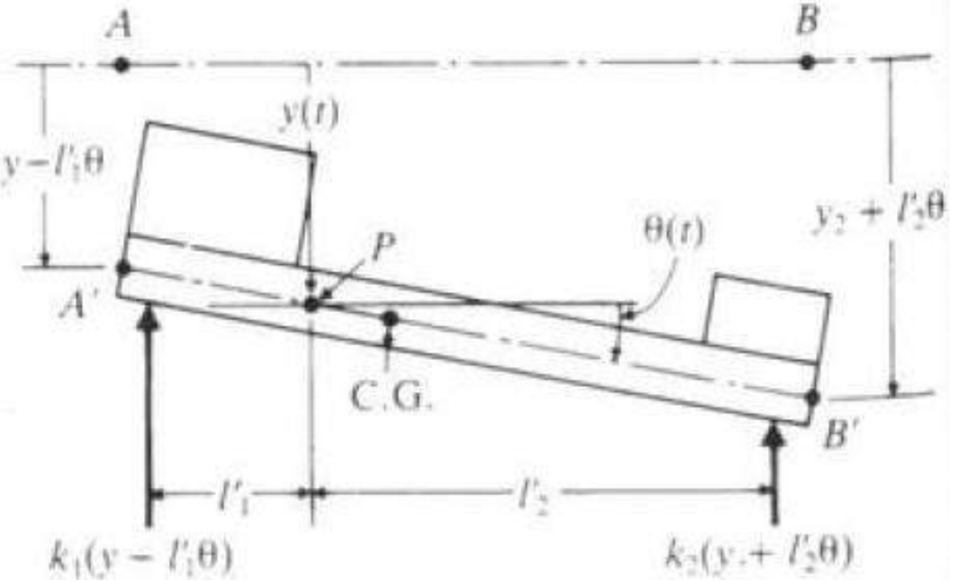
- Using the FBD, the translational and rotational equations of motion are:

$$m\ddot{y} = -k_1(y - l_1'\theta) - k_2(y - l_2'\theta) - me\ddot{\theta} \qquad \text{and} \qquad J_p\ddot{\theta} = k_1(y - l_1'\theta)l_1' - k_2(y - l_2'\theta)l_2' - me\ddot{y}$$

in matrix form:

$$\begin{bmatrix} m & me \\ me & J_p \end{bmatrix}\begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & (k_2l_2' - k_1l_1') \\ (k_2l_2' - k_1l_1') & (k_1l_1'^2 + k_2l_2'^2) \end{bmatrix}\begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- As each eqn. contains both y, y'', θ *and* θ'' the system is coupled with both ***elastic (static) and mass (dynamic) coupling***
- When $k_1l'_1 = k_2l'_2$, the system is **dynamically** coupled **only** → the inertial force my'' produced by vertical motion will induce a rotational motion (my''e) and vice verca.



Co-ordinate coupling

- General case for viscously damped 2DoF:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}+\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- System has elastic (static) coupling if the stiffness matrix is not diagonal
- System has damping or velocity (dynamic) coupling if the damping matrix is not diagonal
- System has mass or inertial (dynamic) coupling if the mass matrix is not diagonal

- The system behaviour does not depend on the choice of coordinates!
- There exists a set of coordinates which will produce (statically and dynamically) uncoupled equations of motions → ***principal*** or ***natural*** coordinates. These uncoupled equations can be solved independently.

UNIT: III

Harmonic Forced Vibrations- Undamped

- **Harmonically forced vibrations – undamped**
- The harmonic excitation forces are:

$$F_1(t) = F_1 \sin(\omega_f t) \quad and \quad F_2(t) = F_2 \sin(\omega_f t)$$

where ω_f is the forcing frequency.

- Applying Newton's 2nd law gives the eqns. of motion:

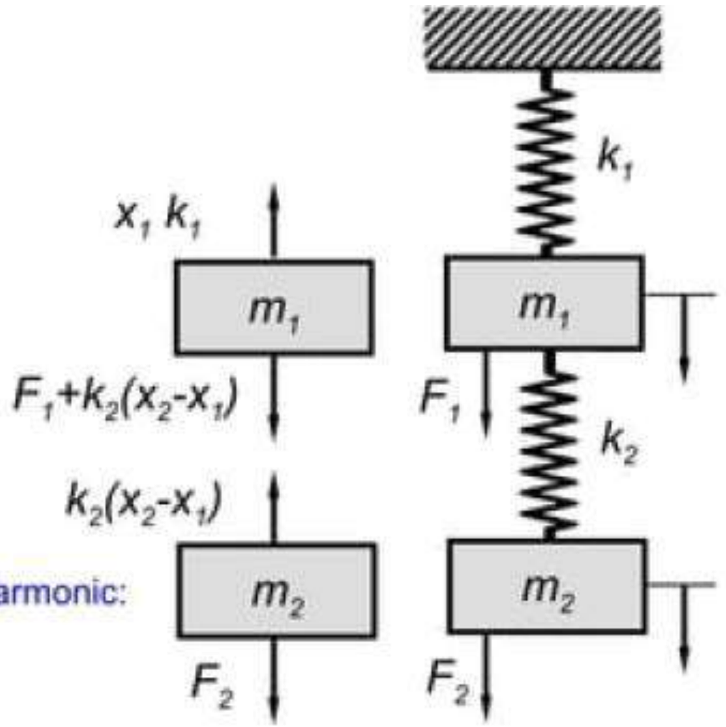
$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F_1 \sin(\omega_f t)$$
$$m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = F_2 \sin(\omega_f t)$$

- Assuming that the solutions will take the form of the excitation – harmonic:

$$x_1 = X_1 \sin(\omega_f t) \quad and \quad x_2 = X_2 \sin(\omega_f t)$$

- Substituting for x_1 and x_2 in the eqns. of motion:

$$(-m_1\omega_f^2 + k_1 + k_2)X_1 \sin(\omega_f t) - k_2X_2 \sin(\omega_f t) = F_1 \sin(\omega_f t)$$
$$(-m_2\omega_f^2 + k_2)X_2 \sin(\omega_f t) - k_2X_1 \sin(\omega_f t) = F_2 \sin(\omega_f t)$$



Harmonic Forced Vibrations- Undamped

Dividing throughout by $\sin(\omega_f t)$ and putting in matrix form :

$$\begin{bmatrix} (k_1 + k_2 - m_1\omega_f^2) & -k_2 \\ -k_2 & (k_2 - m_2\omega_f^2) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

or

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad \rightarrow \quad d_{11}X_1 + d_{12}X_2 = F_1 \quad and \quad d_{21}X_1 + d_{22}X_2 = F_2$$

The response amplitudes X_1 and X_2 can be determined using Cramer's rule:

$$X_1 = \frac{\begin{vmatrix} F_1 & d_{12} \\ F_2 & d_{22} \end{vmatrix}}{\begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}} = \frac{d_{22}F_1 - d_{12}F_2}{d_{11}d_{22} - d_{21}d_{12}} \quad \text{and} \quad X_2 = \frac{\begin{vmatrix} d_{11} & F_1 \\ d_{21} & F_2 \end{vmatrix}}{\begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}} = \frac{d_{11}F_2 - d_{21}F_1}{d_{11}d_{22} - d_{21}d_{12}}$$

- Note: the determinant (characteristic equation) can be equated to zero ($d_{11}d_{22} - d_{21}d_{12} = 0$) to define the system natural frequencies.
- Under forced excitation, when $d_{11}d_{22} - d_{21}d_{12} = 0$ the response amplitudes X_1 and X_2 $\rightarrow \infty$
- This defines resonance conditions (excitation frequency corresponds to either natural frequencies)
- Note: Due to coupling both masses will exhibit resonance when the excitation force is applied to only one mass:

UNIT: III

Harmonic Forced Vibrations- Undamped

- A mass-spring assembly added to a single degree of freedom with a natural frequency ω_n tuned to the forcing frequency ω_f will act as a vibration absorber and reduce the vibration of the main mass to zero.
- Undamped vibration absorbers are designed so that the natural frequencies of the resulting system are displaced away from the excitation frequency.
- The equations of motion of the main mass m_1 and the auxiliary mass m_2 are:

$$m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = F_0 \sin(\omega t)$$
$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$$

Rearranging

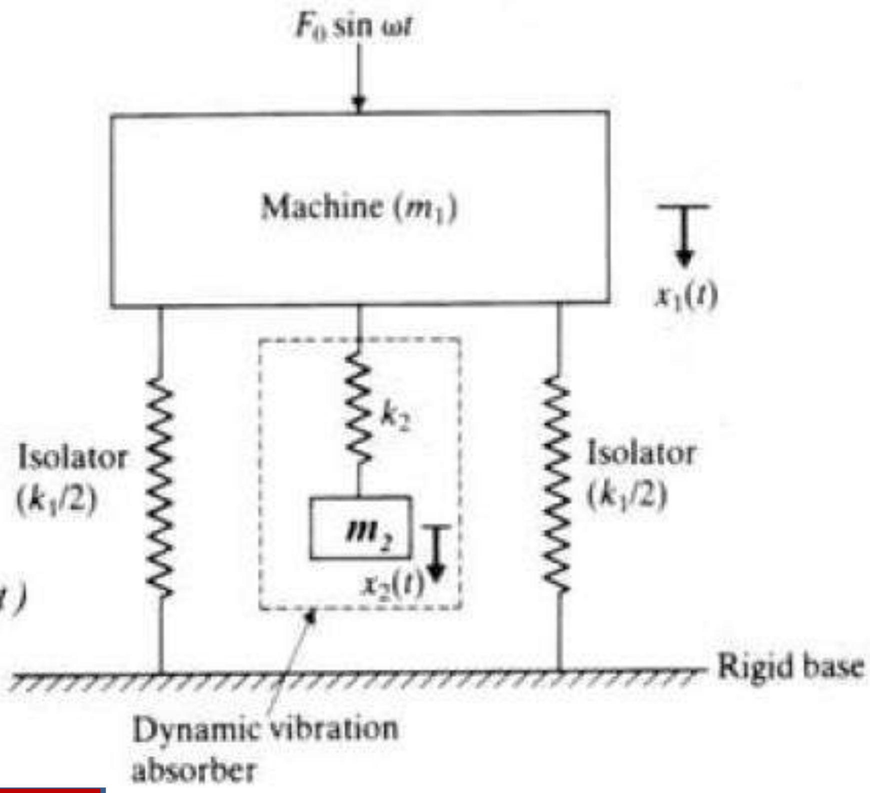
$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F_0 \sin(\omega t)$$
$$m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0$$

Assuming harmonic solutions

$$x_j(t) = X_j \sin(\omega t) \qquad j = 1, 2$$

And substituting into the eqns. of motion:

$$\left[-\omega^2 m_1 X_1 + (k_1 + k_2)X_1 - k_2X_2\right]\sin(\omega t) = F_0 \sin(\omega t)$$
$$-\omega^2 m_2X_2 + k_2X_2 - k_2X_1 = 0$$



Harmonic Forced Vibrations- Undamped

In matrix form :

$$\begin{bmatrix} -\omega^2 m_1 + (k_1 + k_2) & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

Using Cramer's rule to determine the response amplitudes X_1 and X_2 :

$$X_1 = \frac{\begin{vmatrix} F_1 & d_{12} \\ F_2 & d_{22} \end{vmatrix}}{\begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}} = \frac{d_{22}F_1 - d_{12}F_2}{d_{11}d_{22} - d_{21}d_{12}} \quad \text{and} \quad X_2 = \frac{\begin{vmatrix} d_{11} & F_1 \\ d_{21} & F_2 \end{vmatrix}}{\begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}} = \frac{d_{11}F_2 - d_{21}F_1}{d_{11}d_{22} - d_{21}d_{12}}$$

Or

$$X_1 = \frac{(k_2 - \omega^2 m_2) F_0}{(k_1 + k_2 - \omega^2 m_1)(k_2 - \omega^2 m_2) - k_2^2} \quad \text{and} \quad X_2 = \frac{k_2 F_0}{(k_1 + k_2 - \omega^2 m_1)(k_2 - \omega^2 m_2) - k_2^2}$$

- In order to minimise the amplitude of mass 1, the numerator of X_1 should be equated to zero which produces:

$$\omega^2 = \frac{k_2}{m_2}$$

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Harmonic Forced Vibrations- Undamped

If the original machine was operating near resonance :

$$\omega^2 \; ; \; \omega_1^2 = \frac{k_1}{m_1}$$

If the absorber is designed so that its natural frequency corresponds to the forcing frequency :

$$\omega^2 = \frac{k_2}{m_2} = \frac{k_1}{m_1}$$

The amplitude of the machine (m_1) at its original resonant frequency will be zero.

Since

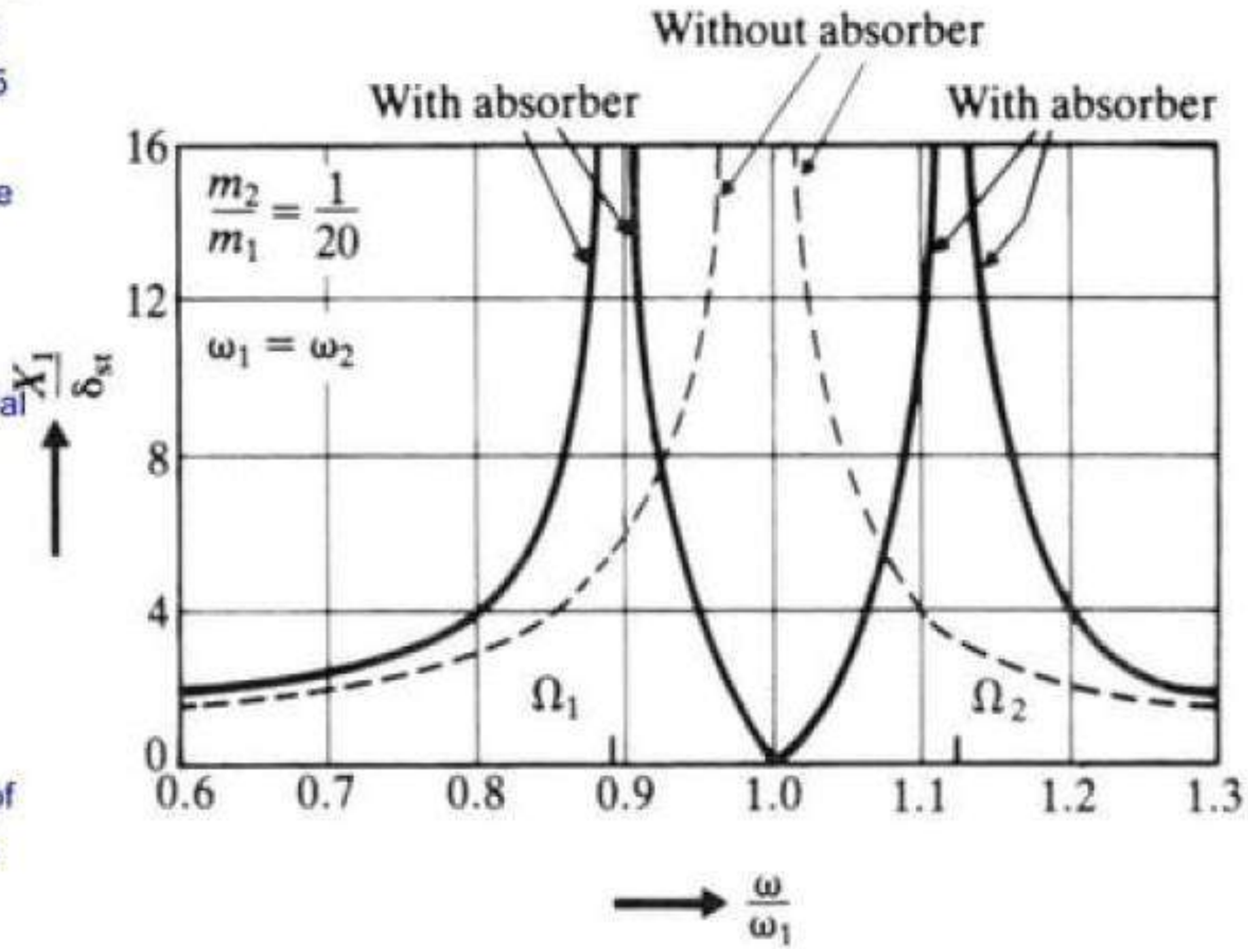
$$\delta_{st} = \frac{F_0}{k_1}, \quad \omega_1 = \sqrt{\frac{k_1}{m_1}} \quad and \quad \omega_2 = \sqrt{\frac{k_2}{m_2}}$$

The dynamic response (magnification factor) of the main mass and the auxiliary mass (absorber) are :

$$\frac{X_1}{\delta_{st}} = \frac{1-\left(\frac{\omega}{\omega_2}\right)^2}{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_1}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_2}\right)^2\right]-\frac{k_2}{k_1}} \quad \text{and} \quad \frac{X_2}{\delta_{st}} = \frac{1}{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_1}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_2}\right)^2\right]-\frac{k_2}{k_1}}$$

Harmonic Forced Vibrations- Undamped Absorber

- The size of the auxiliary mass m_2 is governed by the allowable deflection X_2.
- These systems can be quite effective over a reasonable frequency band ± 5 %.
- The new system has an added degree of freedom hence two resonance peaks.
- The system will pass thru the first resonance during startup, it is essential that the run-up time is minimised.
- Otherwise, introduce damping to prevent large vibrations of m_1 if the excitation frequency is likely to vary.
- At $\omega = \omega_1$, $X_1 = 0$ and $X_2 = -k_1 \delta_{st}/k_2 = -F_0/k_2$ which shows that the force exerted by the absorber mass is out of phase with (counteracts) the exciting force which causes X_1 to reduce to zero.



UNIT: III

Harmonic Forced Vibrations- Undamped Absorber

- Harmonically forced vibrations – damped absorber

- Introducing a viscous damper produces the following eqns. of motion:

$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) = F_0 \sin(\omega t)$$
$$m_2\ddot{x}_2 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) = 0$$

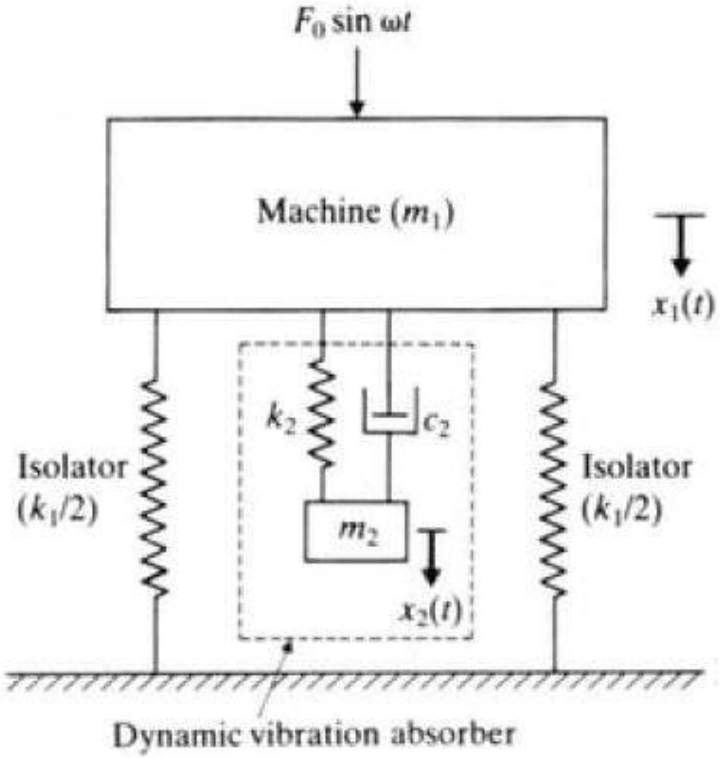
Assuming harmonic solutions in the form :

$$x_j(t) = X_j e^{i\omega t} \qquad j = 1, 2$$

Yields the steady-state amplitudes:

$$X_1 = \frac{F_0\left(k_2 - \omega^2 m_2 + ic_2\omega\right)}{\left[\left(k_1 - \omega^2 m_1\right)\left(k_2 - \omega^2 m_2\right) - m_2 k_2 \omega^2\right] + ic_2\omega\left(k_1 - \omega^2 m_1 - \omega^2 m_2\right)}$$

$$X_2 = \frac{X_1\left(k_2 + ic_2\omega\right)}{\left(k_2 - \omega^2 m_2 + ic_2\omega\right)}$$



UNIT: III

Harmonic Forced Vibrations- Damped Absorber

Using the following definitions :

Mass ratio :	$\mu = m_2 / m_1$
Static deflection :	$\delta_{st} = F_0 / k_1$
Square absorber natural frequency :	$\omega_a^2 = k_2 / m_2$
Square main mass natural frequency :	$\omega_n^2 = k_1 / m_1$
Natural frequency ratio :	$f = \omega_a / \omega_n$
Forced frequency ratio :	$g = \omega / \omega_n$
Critical damping constant :	$c_c = 2 m_2 \omega / \omega_n$
Damping ratio :	$\zeta = c_2 / c_c$

The magnitude ratios can be written as :

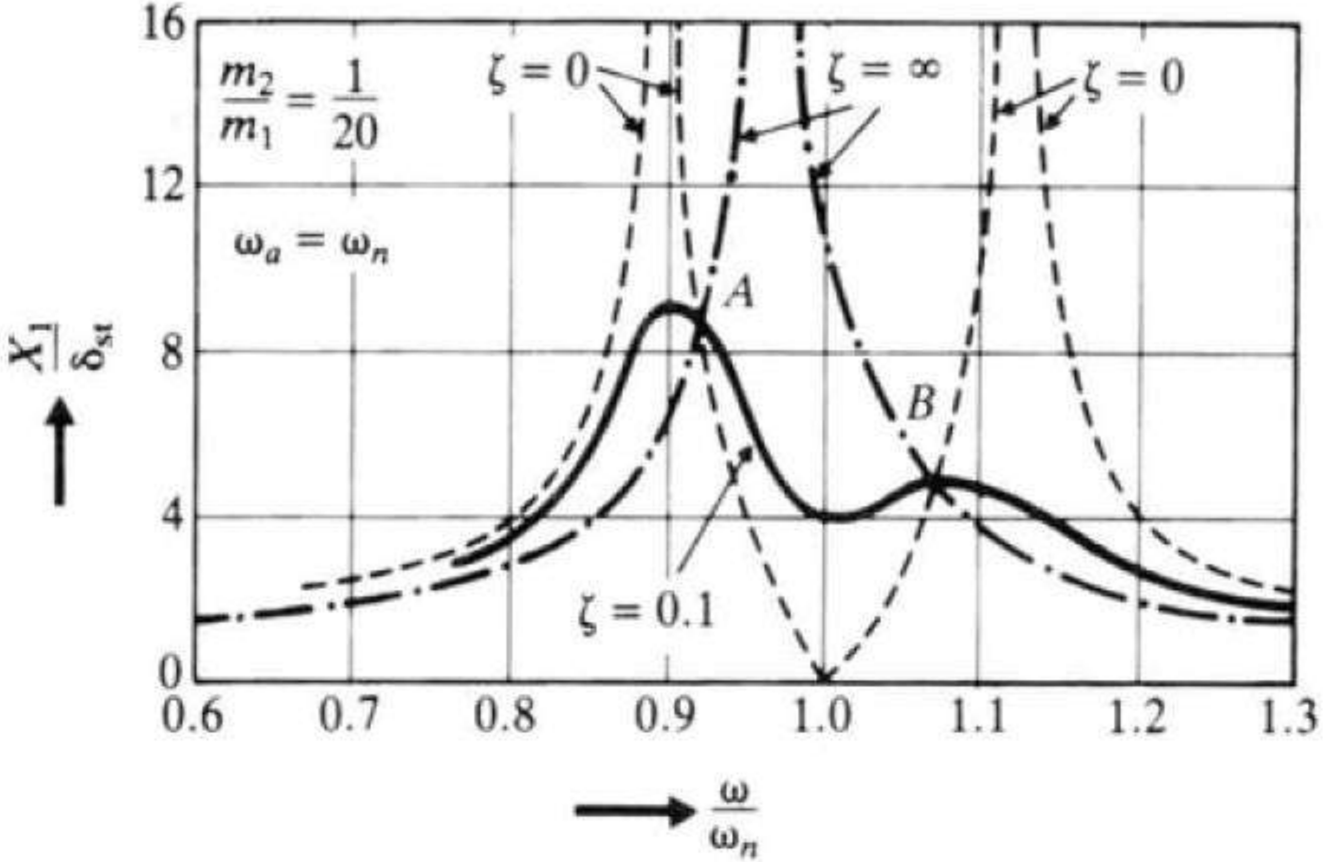
$$\frac{X_1}{\delta_{st}} = \sqrt{\frac{(2\zeta g)^2 + (g^2 - f^2)^2}{(2\zeta g)^2 (g^2 - 1 + \mu g^2)^2 + \{\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)\}^2}}$$

$$\frac{X_2}{\delta_{st}} = \sqrt{\frac{(2\zeta g)^2 + f^4}{(2\zeta g)^2 (g^2 - 1 + \mu g^2)^2 + \{\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)\}^2}}$$

UNIT: III

Harmonic Forced Vibrations- Damped Absorber

$$\frac{X_1}{\delta_{st}} = \sqrt{\frac{(2\zeta g)^2 + (g^2 - f^2)^2}{(2\zeta g)^2 (g^2 - 1 + \mu g^2)^2 + \{\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)\}^2}}$$



Harmonic Forced Vibrations- Damped Absorber

- When damping is infinite, the two masses are rigidly coupled and the system behaves as an undamped single DoF system with mass $m_1 + m_2$ and stiffness k_1
- X_1 approaches ∞ when $\zeta = 0$ and $\zeta = \infty$
- The amplitude of the absorber mass is always greater that that of the main mass. Allow for large vibration amplitudes and consider fatigue issues for design of absorber springs.
- X_1 will have a minimum
- All damping values produce curves which intersect at **A** and **B**
- The frequencies of A and B can be located by substituting the extreme conditions $\zeta = 0$ and $\zeta = \infty$ into the magnitude ratio equation.
- It has been shown that vibration absorbers operate optimally when the ordinates of A and B are equal for which:

$$f = \omega_a / \omega_n = \frac{1}{(1+\mu)} = \frac{1}{(1+m_2/m_1)}$$

- Such systems are known as ***tuned vibration absorbers***.