MECHANICAL VIBRATIONS

Course Name: B.Tech-ME

Semester: 7th

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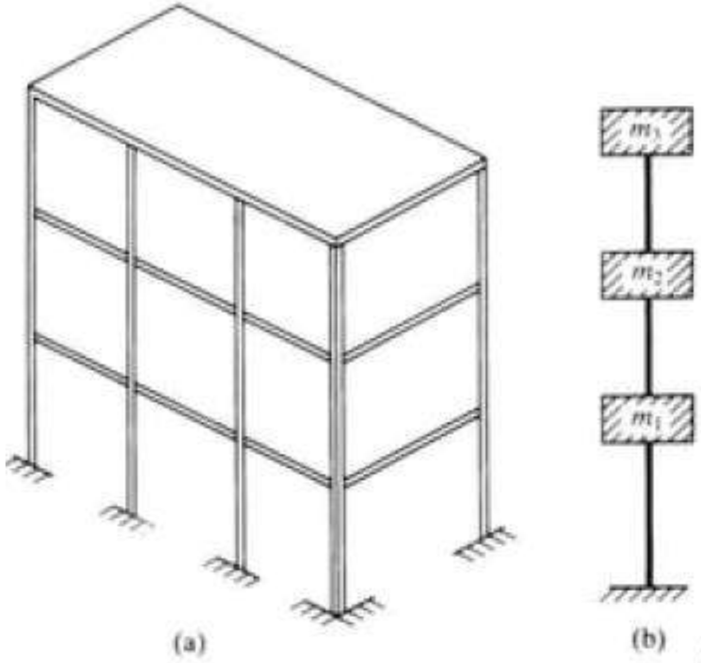
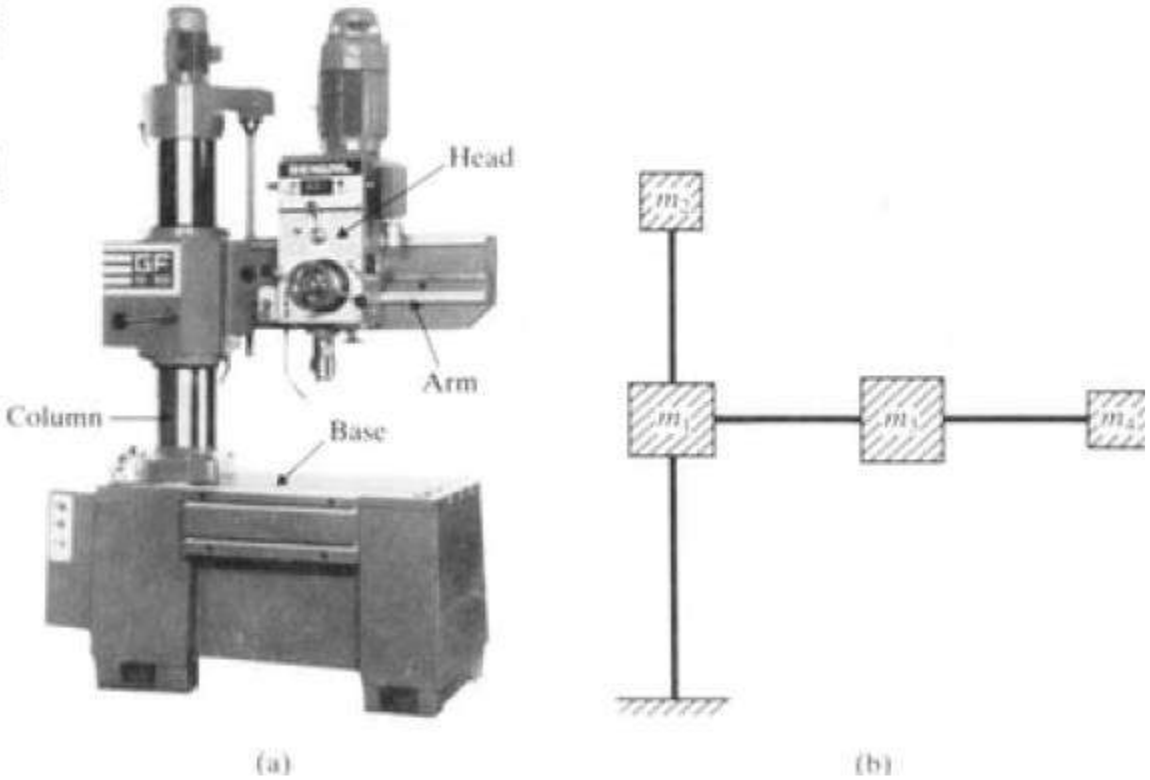
UNIT: IV Multi Degree of Freedom Systems

- Vibration analysis of continuous systems require solution to partial differential equations which do not always exist
- Analysis of multi DoF systems requires solution of a collection of ordinary differential equations.
- Continuous systems are often approximated by MDoF systems.
- Previous principles apply:
  - One eqn. of motion for each degree of freedom
  - One generalised coordinate for each degree of freedom
  - The number of natural frequencies and mode shapes are equal to the number of DoFs
  - The natural frequencies are determined by equating the determinant to zero (solution to characteristic equations becomes more complex as number of DoF increases)
- Eqns. of motion obtained from Newton's second law, influence coefficients or Lagrange's equations.

- Modelling continuous systems as MDoF systems:
  - ***Finite element models:***
    - The geometry of a distributed mass system is replaced by a large number of small structural elements (m,c,k)
    - A simple solution is assumed for each element
    - Inter-element compatibility and equilibrium is used to approximate the solution
  - ***Lumped-mass or discrete-mass models:***
    - The (distributed) mass or inertia of the system is replaced by a finite number of rigid bodies (lumped mass)
    - These lumped mass are connected by mass-less spring and damping elements.
    - Linear or angular coordinates are used to describe the motion of each lumped mass element
    - Better accuracy is usually achieved when more lumped masses are used

Lumped-mass or Discrete mass model

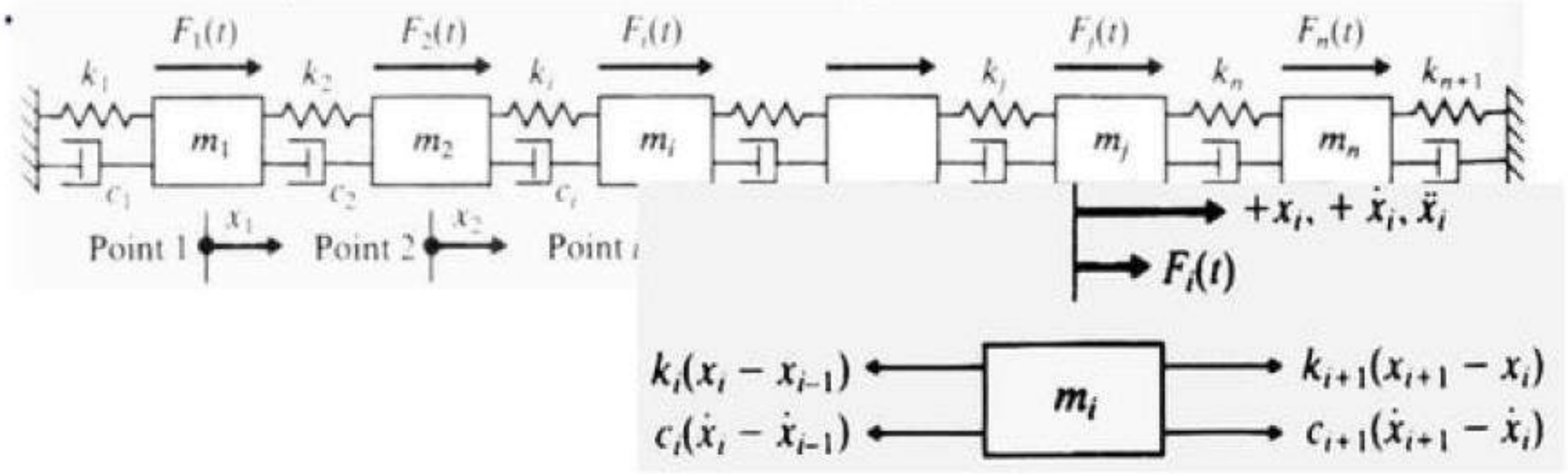




Equation of Motion

1. Define suitable coordinates to describe the position of each lumped mass in the model
2. Establish the static equilibrium of the system and determine the displacement of each lumped mass wrt to their respective static equilibrium position.
3. Draw the free-body diagram for each lumped mass in the model. Indicate the spring, damping and external forces on each mass element when a positive displacement and velocity is applied to each mass element.
4. Generate the equation of motion for each mass element by applying Newton's second law of motion with reference to the free-body diagrams:

$$m_i \ddot{x}_i = \sum_j F_{ij} \quad (\text{for mass } m_i) \quad \text{and} \quad J_i \ddot{\theta}_i = \sum_j M_{ij} \quad (\text{for rigid body of inertia } J)$$



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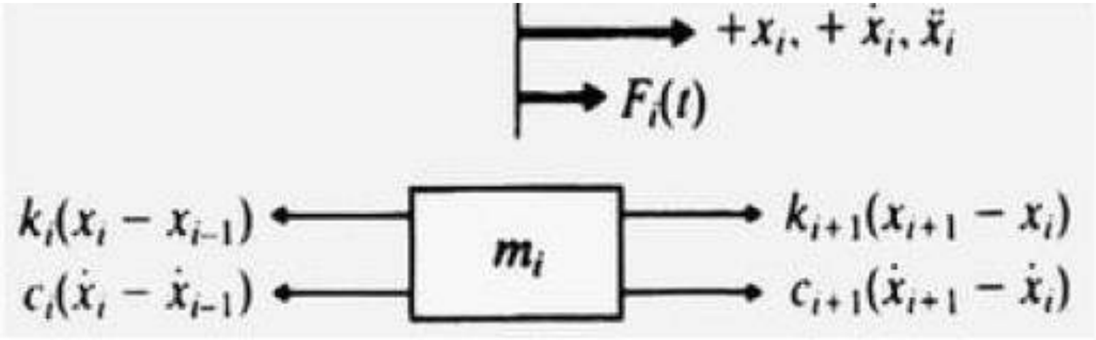
Equation of Motion



$$m_i\ddot{x}_i = -k_i(x_i - x_{i-1}) + k_{i+1}(x_{i+1} - x_i) - c_i(\dot{x}_i - \dot{x}_{i-1}) + c_{i+1}(\dot{x}_{i+1} - \dot{x}_i) + F_i \quad for\ i = 1,2,3\ldots,n-1$$

Rearranging:

$$m_i\ddot{x}_i - c_i\dot{x}_{i-1} + (c_i + c_{i+1})\dot{x}_i - c_{i+1}\dot{x}_{i+1} - k_i x_{i-1} + (k_i + k_{i+1})x_i - k_{i+1}x_{i+1} = F_i \quad for\ i = 1,2,3\ldots,n-1$$

- Note that the system has both stiffness and damping coupling
- The equations of motion of masses m_1 and m_n at the extremities of the system are obtained by setting $i = 1$ & $x_{i-1} = 0$ and $i = n$ & $x_{n+1} = 0$

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1$$

$$m_n\ddot{x}_n - c_n\dot{x}_{n-1} + (c_n + c_{n+1})\dot{x}_n - k_n x_{n-1} + (k_n + k_{n+1})x_n = F_n$$

- In matrix form:

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F}$$

Equation of Motion

- Where the mass matrix [*m*], the damping matrix [*c*] and the stiffness matrix [*k*] are given by:

$$[m] = \begin{bmatrix} m_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & m_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & m_3 & \dots & 0 & 0 \\ . & & & & & . \\ . & & & & & . \\ . & & & & & . \\ 0 & 0 & 0 & \dots & 0 & m_n \end{bmatrix}$$

$$[c] = \begin{bmatrix} (c_1+c_2) & -c_2 & 0 & \dots & 0 & 0 \\ -c_2 & (c_2+c_3) & -c_3 & \dots & 0 & 0 \\ 0 & -c_3 & (c_3+c_4) & \dots & 0 & 0 \\ . & & & & . & . \\ . & & & & . & . \\ . & & & & . & . \\ 0 & 0 & 0 & \dots & -c_n & (c_n+c_{n+1}) \end{bmatrix}$$

Equation of Motion

$$[k] = \begin{bmatrix} (k_1+k_2) & -k_2 & 0 & \cdots & 0 & 0 \\ -k_2 & (k_2+k_3) & -k_3 & \cdots & 0 & 0 \\ 0 & -k_3 & (k_3+k_4) & \cdots & 0 & 0 \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -k_n & (k_n+k_{n+1}) \end{bmatrix}$$

- And the displacement. Velocity, acceleration and excitation force vectors are given by:

$$\vec{x} = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{Bmatrix} \quad \dot{\vec{x}} = \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n(t) \end{Bmatrix} \quad \ddot{\vec{x}} = \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_n(t) \end{Bmatrix} \quad \vec{F} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ \cdot \\ \cdot \\ \cdot \\ F_n(t) \end{Bmatrix}$$

Equation of Motion

- In general terms:

$$[m] = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{21} & m_{22} & m_{23} & \cdots & m_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ m_{n1} & m_{n1} & m_{n3} & \cdots & m_{nn} \end{bmatrix} \quad [c] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{bmatrix} \quad [k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ k_{n1} & k_{n2} & k_{n3} & \cdots & k_{nn} \end{bmatrix}$$

Influence Co-efficient

- It is sometimes practical to express the eqns. of motion of MDoF systems in terms of ***influence coefficients***
- The elements of the stiffness matrix are known as the ***stiffness*** influence coefficients and relate the force at a point in the system with the displacement applied at another point in the system.
- The stiffness influence coefficient k_{ij} is defined as the force at point *i* due to a unit displacement at point *j* when all other points, except *j*, are fixed.
- The total force at *i* is the sum of the forces due to all applied displacements.:

$$F_i = \sum_{j=1}^{n} k_{ij} x_j \qquad i = 1,2,3...n \qquad \text{or} \qquad \vec{F} = [k]\,\vec{x} \qquad \text{where} \qquad [k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ k_{n1} & k_{n2} & k_{n3} & \cdots & k_{nn} \end{bmatrix}$$

Influence Co-efficient: Stiffness

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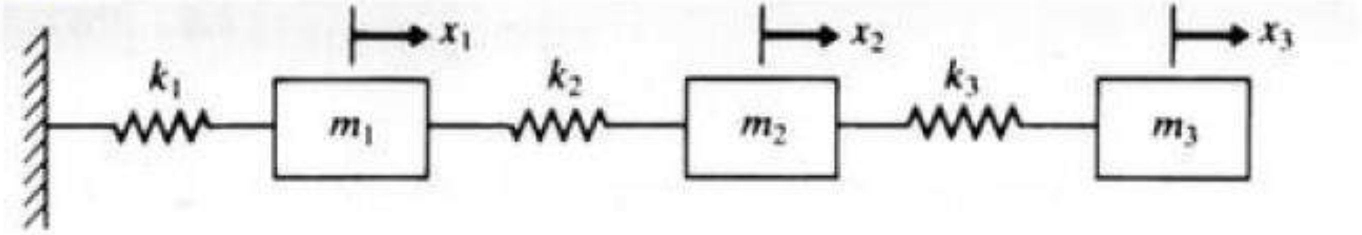
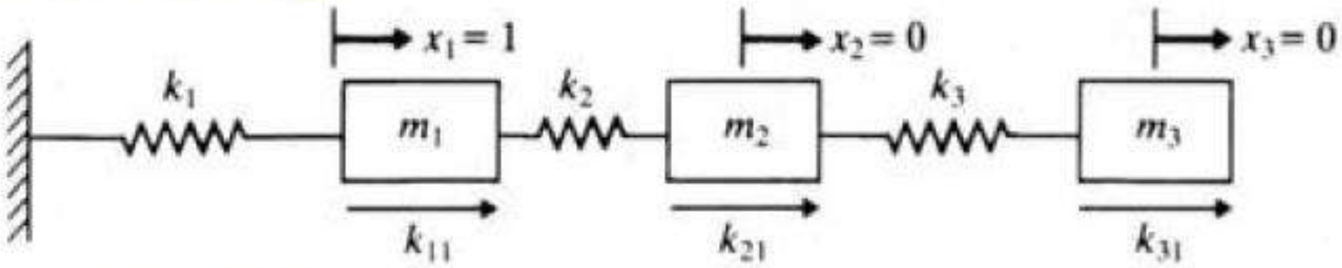
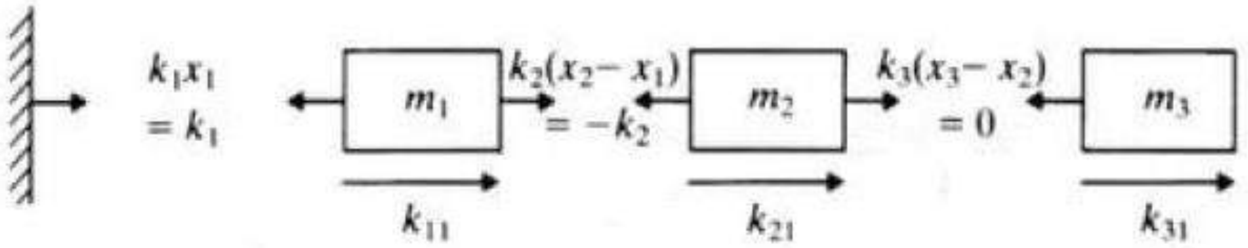
- Example:



- Use static equilibrium to determine the stiffness influence coefficients.
- Step 1: $x_1 = 1$, $x_2 = 0$, $x_3 = 0$.



- For which the free–body diagram is:



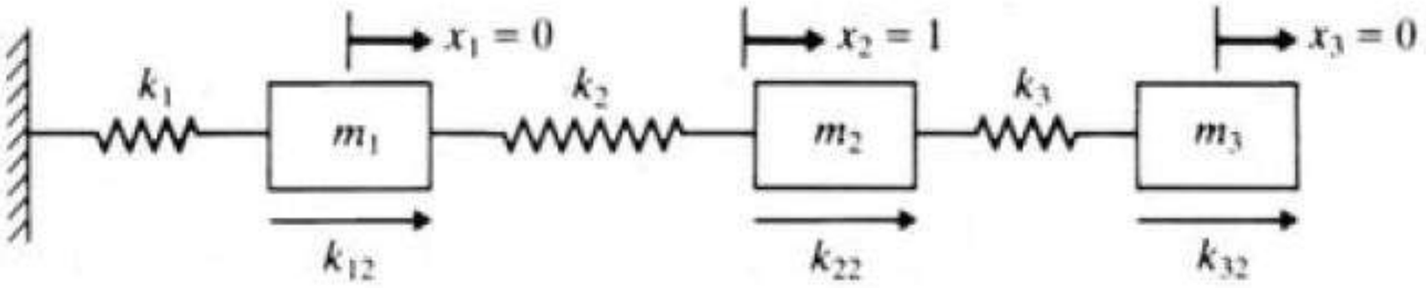
$k_{11} = F_1/x_{11} = F_1 = k_1 + k_2$

$k_{21} = -k_2$

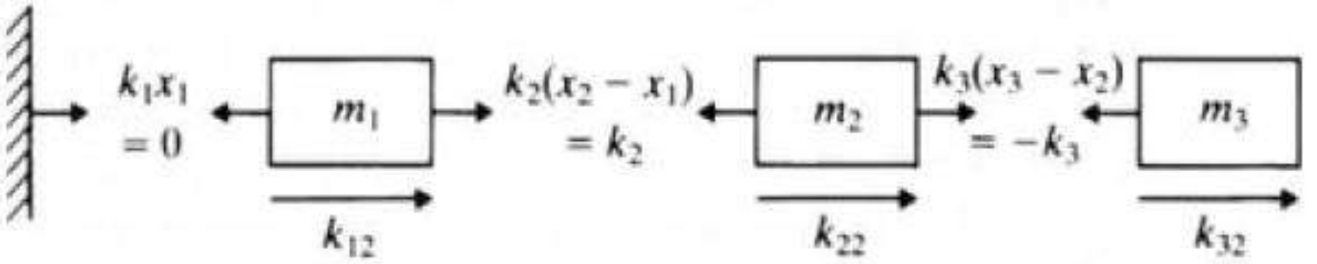
$k_{31} = 0$

Influence Co-efficient: Stiffness

- Step 2: $x_1 = 0$, $x_2 = 1$, $x_3 = 0$.



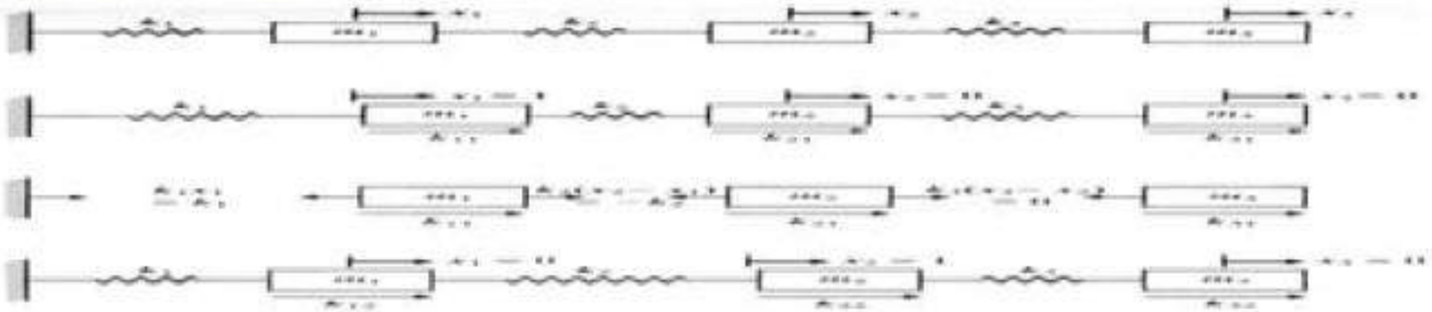
- For which the free–body diagram is:



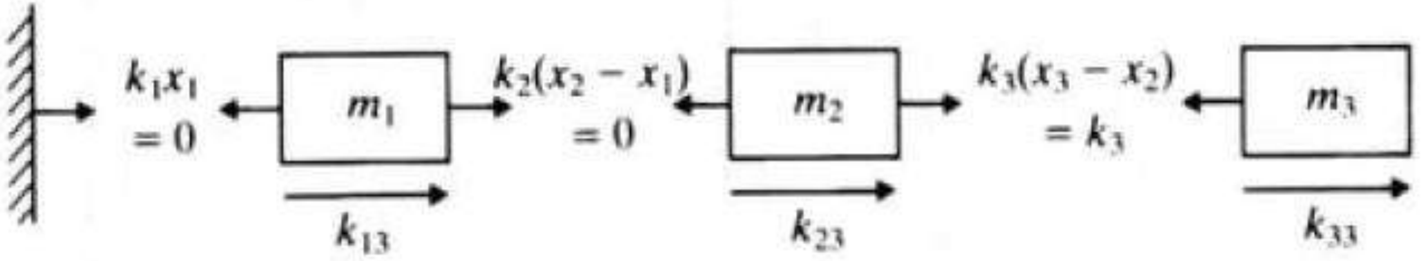
$$k_{12} = -k_2$$
$$k_{22} = k_2 + k_3$$
$$k_{32} = -k_3$$

Influence Co-efficient: Stiffness

- Step 3: $x_1 = 0,\ x_2 = 0,\ x_3 = 1.$



- For which the free–body diagram is:



$k_{13} = 0$

$k_{23} = -k_3$

$k_{33} = k_3$

Influence Co-efficient: Stiffness

- The system stiffness matrix is:

$$[k]=\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}=\begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

- The calculation of n stiffness influence coefficients require the solution of n simultaneous equations.
- Thus the computation of stiffness influence coefficients for a system with n degrees of freedom may require a significant effort (up to n^2 computations)

Influence Co-efficient: Flexibility

- It is sometimes easier to define the system in terms of the ***flexibility influence coefficients***
- The flexibility influence coefficients relates the displacement at a point in the system with the force applied at another point in the system.
- The flexibility influence coefficient a_{ij} is defined as the deflection at point *i* due to a unit force point *j* with no other forces acting on the system.
- For a linear system:

$$x_{ij} = a_{ij} F_j$$

- When several forces act at various points in the system, F_j *for j* = 1, 2, 3....*n*, the total deflection at point *i* is the sum of the deflections caused by each individual applied force:

$$x_i = \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} a_{ij} F_j \qquad i = 1,2,3.....n \qquad \text{in matrix form:} \qquad \vec{x} = [a]\,\vec{F}$$

where $\vec{x}$ and $\vec{F}$ are the displacement and force vectors and $[a]$ is the flexibility matrix:

$$[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

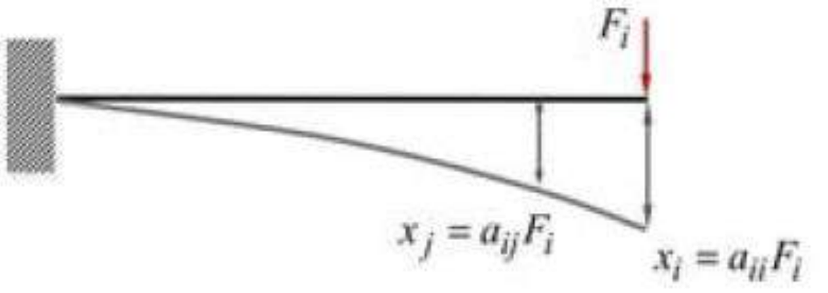
Influence Co-efficient: Flexibility

- Not unexpected that the flexibility matrix is related to the stiffness matrix.

$$[a]^{-1}\vec{x} = [a]\vec{F}[a]^{-1}$$

$$\vec{F} = [a]^{-1}\vec{x} = [k]\vec{x}$$

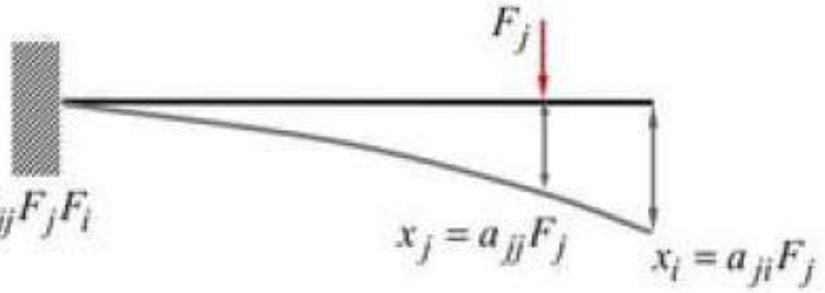
$$[a]^{-1} = [k]$$



- **Reciprocity theorem:** For a linear system : $a_{ij} = a_{ji}$
- Consider the work done by forces F_i andf F_j

Case 1: $W_i = \frac{1}{2}F_i x_i = \frac{1}{2}a_{ii}F_i^2$

Case 2: $W_j = \frac{1}{2}F_j x_j = \frac{1}{2}a_{jj}F_j^2$



When F_i and F_j are applied sequentially the total work is:

$$W_{ij} = \frac{1}{2}a_{ii}F_i^2 + \frac{1}{2}a_{jj}F_j^2 + x_jF_i = \frac{1}{2}a_{ii}F_i^2 + \frac{1}{2}a_{jj}F_j^2 + a_{ij}F_jF_i$$

and when F_j is applied before F_i the total work is:

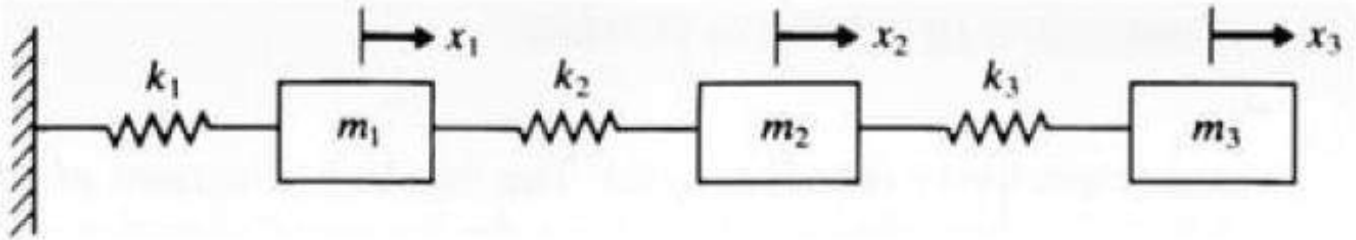
$$W_{ji} = \frac{1}{2}a_{jj}F_j^2 + \frac{1}{2}a_{ii}F_i^2 + x_iF_j = \frac{1}{2}a_{ii}F_i^2 + \frac{1}{2}a_{jj}F_j^2 + a_{ji}F_iF_j$$

Since the total work done is not dependent on the sequence of applied force :

$$W_{ij} = W_{ji} \qquad \text{hence} \qquad a_{ij} = a_{ji}$$

Influence Co-efficient: Flexibility

- Example: Use static equilibrium to determine the flexibility matrix of the system.



- Step 1: Apply a unit load at point 1 only and calculate the deflections of each mass due to the unit load at 1.

$$a_{11} = x_{11} / F_1 = x_{11}$$

Mass 1:

$$k_1 a_{11} = k_2(a_{21} - a_{11}) + F_1$$

$$k_1 a_{11} = k_2(a_{21} - a_{11}) + 1$$

Mass 2:

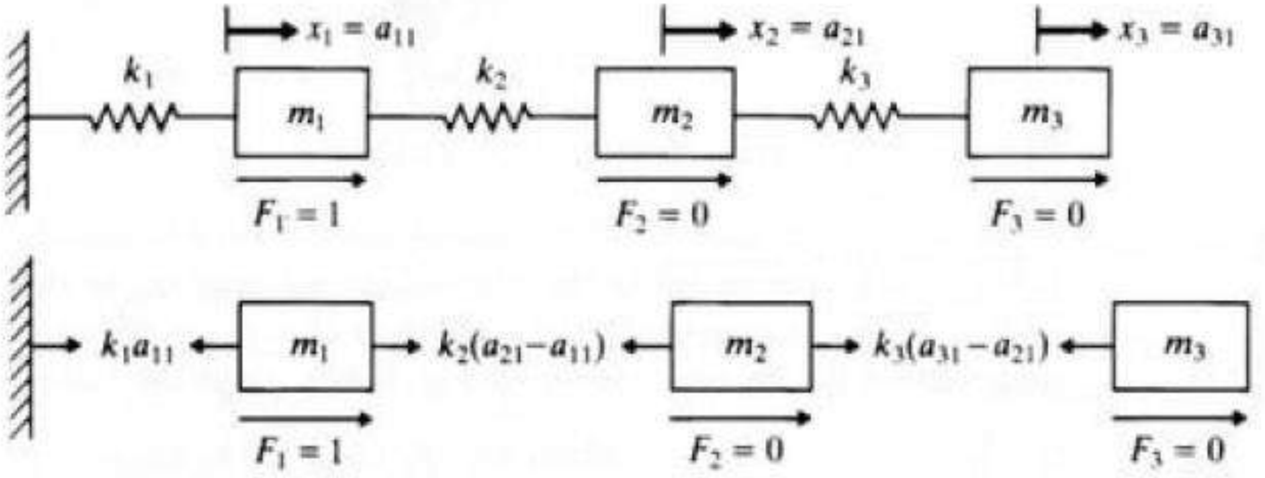
$$k_2(a_{21} - a_{11}) = k_3(a_{31} - a_{21})$$

Mass 3:

$$k_3(a_{31} - a_{21}) = 0$$

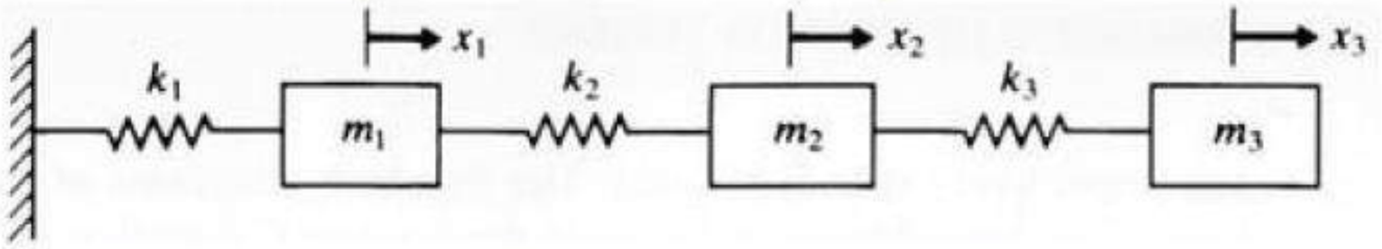
Solving:

$$a_{11} = \frac{1}{k_1},\ a_{21} = \frac{1}{k_1},\ a_{31} = \frac{1}{k_1}.$$



Influence Co-efficient: Flexibility

- Example: Use static equilibrium to determine the flexibility matrix of the system.



- Step 2: Apply a unit load at point 2 only and calculate the deflections of each mass due to the unit load at 2.

Mass 1:

$$k_1 a_{12} = k_2(a_{22} - a_1)$$

Mass 2 :

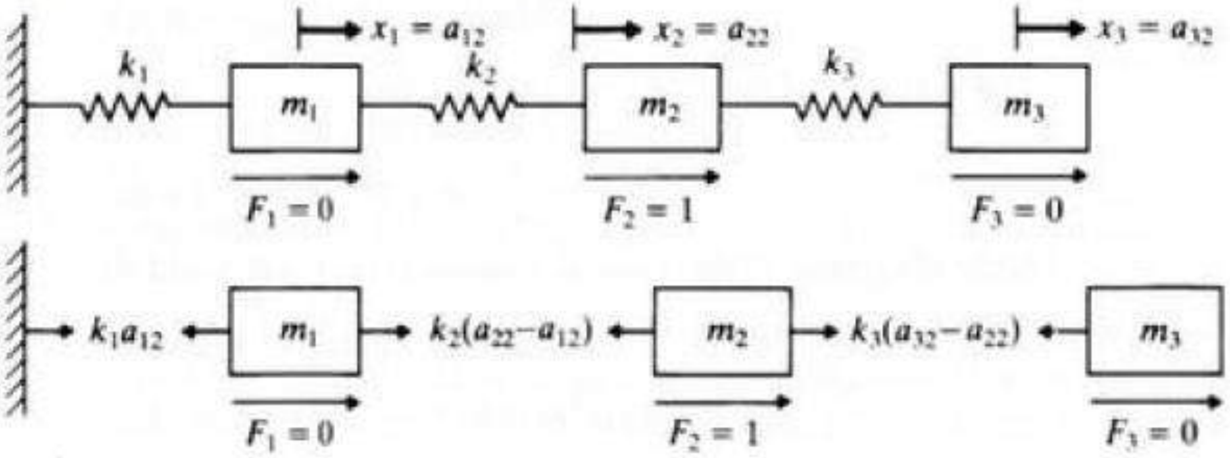
$$k_2(a_{22} - a_{13}) = k_3(a_{32} - a_{22}) + 1$$

Mass 3 :

$$k_3(a_{32} - a_{22}) = 0$$

Solving :

$$a_{12} = \frac{1}{k_1},\ a_{22} = \frac{1}{k_1} + \frac{1}{k_2},\ a_{32} = \frac{1}{k_1} + \frac{1}{k_2}$$



Influence Co-efficient: Flexibility

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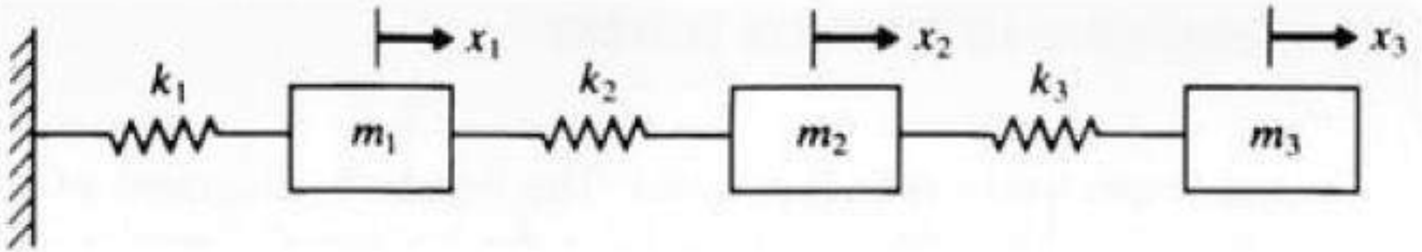
- Example: Use static equilibrium to determine the flexibility matrix of the system.



- Step 3: Apply a unit load at point 3 only and calculate the deflections of each mass due to the unit load at 3.

Mass 1:

$k_1 a_{13} = k_2(a_{23} - a_3)$

Mass 2 :

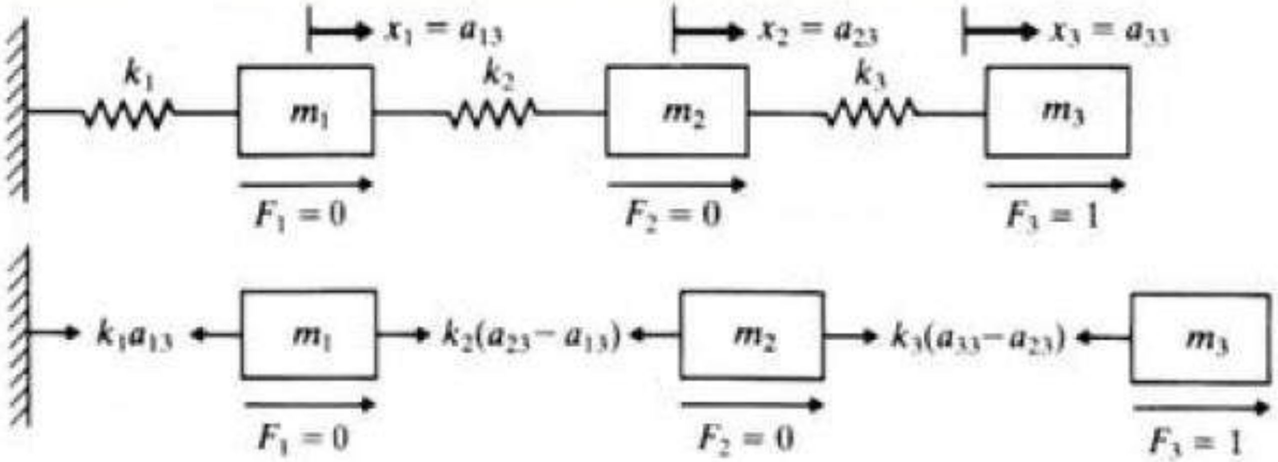
$k_2(a_{23} - a_{13}) = k_3(a_{33} - a_{23})$

Mass 3 :

$k_3(a_{33} - a_{23}) = 1$

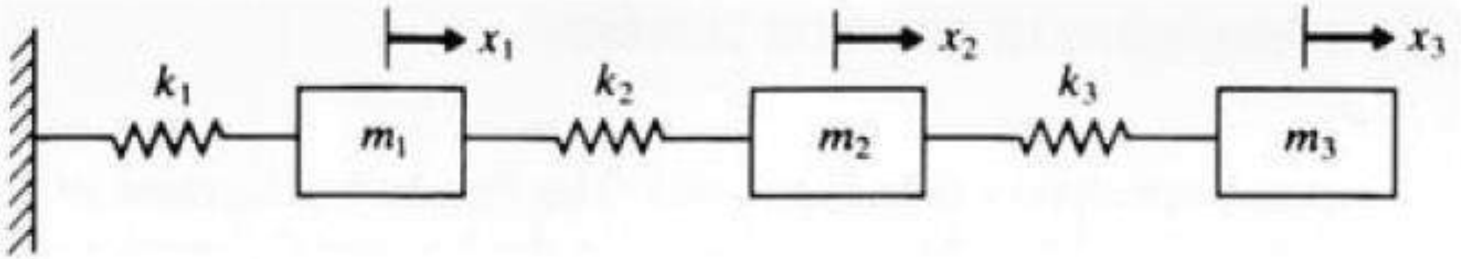
Solving :

$$a_{13} = \frac{1}{k_1},\ a_{23} = \frac{1}{k_1} + \frac{1}{k_2},\ a_{33} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$



Influence Co-efficient: Flexibility

- Example: Use static equilibrium to determine the flexibility matrix of the system.



- The flexibility matrix of the system is:

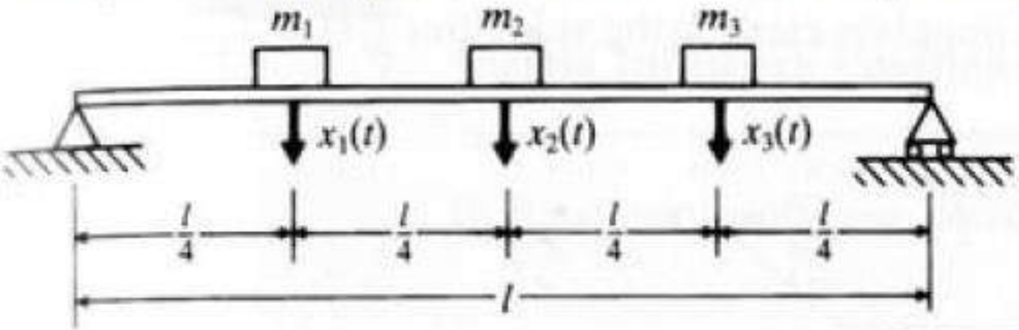
$$[a]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & k_{32} & a_{33} \end{bmatrix}=\begin{bmatrix} 1/k_1 & 1/k_1 & 1/k_1 \\ 1/k_1 & (1/k_1+1/k_2) & (1/k_1+1/k_2) \\ 1/k_1 & (1/k_1+1/k_2) & (1/k_1+1/k_2+1/k_3) \end{bmatrix}$$

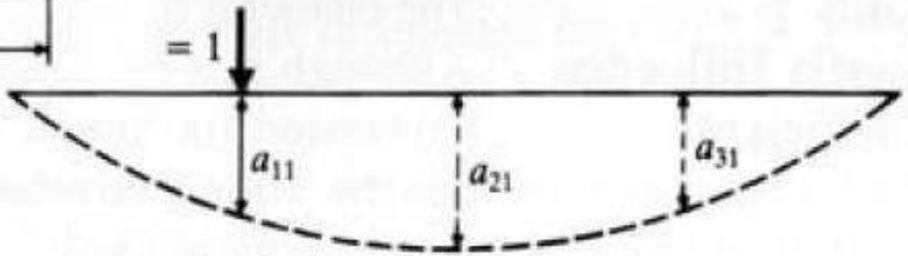
- It can be verified that the inverse of this flexibility matrix is the system stiffness matrix:

$$[k]=\begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

Influence Co-efficient: Flexibility

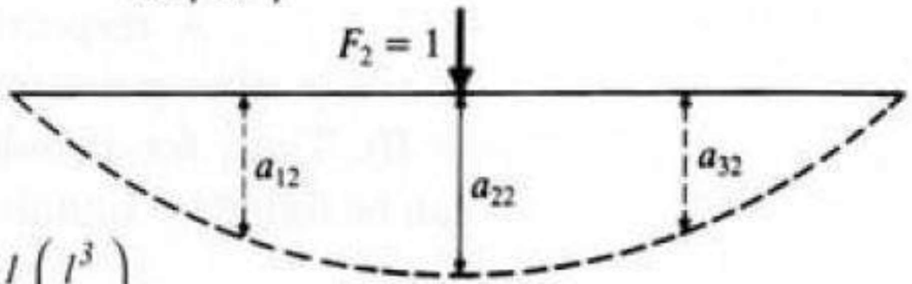
- Example: Use static equilibrium to determine the flexibility matrix of the system.





- Step 1: Apply a unit load at point 1 only and calculate the deflections at points 1, 2 and 3 due to the unit load at 1.

$$a_{11} = x_{11} / F_1 = x_{11} = \frac{9}{768}\left(\frac{l^3}{EI}\right) \qquad a_{12} = \frac{11}{768}\left(\frac{l^3}{EI}\right) \qquad a_{13} = \frac{7}{768}\left(\frac{l^3}{EI}\right)$$

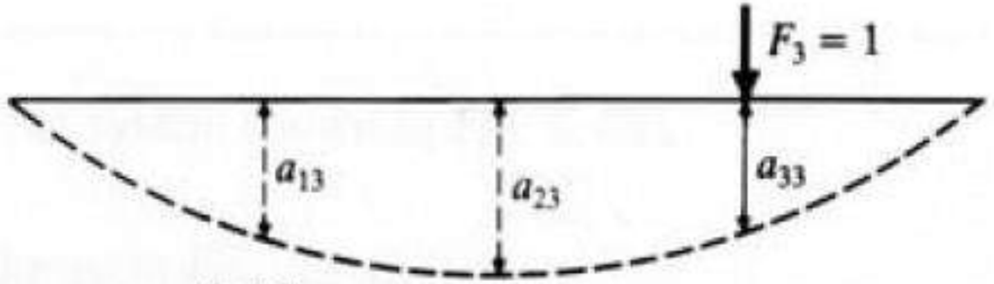


- Step 2: Apply a unit load at point 2 only and calculate the deflections at points 1, 2 and 3 due to the unit load at 2.

$$a_{21} = a_{12} = \frac{11}{768}\left(\frac{l^3}{EI}\right) \qquad a_{22} = \frac{1}{48}\left(\frac{l^3}{EI}\right) \qquad a_{23} = \frac{11}{768}\left(\frac{l^3}{EI}\right)$$

Influence Co-efficient: Flexibility

- **Influence coefficients - flexibility.**
- Step 3: Apply a unit load at point 3 only and calculate the deflections at points 1, 2 and 3 due to the unit load at 3.



$$a_{31} = a_{13} = \frac{7}{768}\left(\frac{l^3}{EI}\right) \qquad a_{32} = a_{23} = \frac{11}{48}\left(\frac{l^3}{EI}\right) \qquad a_{33} = \frac{9}{768}\left(\frac{l^3}{EI}\right)$$

- The system flexibility matrix is:

$$[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & k_{32} & a_{33} \end{bmatrix} = \frac{l^3}{768EI}\begin{bmatrix} 9 & 11 & 7 \\ 11 & 16 & 11 \\ 7 & 11 & 9 \end{bmatrix}$$

Eigen values and Eigen vectors

- The solution to the eqn. of motion of a free undamped MDoF system

$$[m]\ddot{\vec{x}}+[k]\vec{x}=0$$

- defines the (steady-state) harmonic vibration of the system due to an initial disturbance (initial conditions).
- The solution is established by assuming a solution in the form:

$$x_i(t)=X_iT(t) \qquad i=1,2,3.....n$$

where X_i is a constant and T is a function of time.

The amplitude ratio of any two coordinates $\left\{\frac{x_i(t)}{x_j(t)}\right\}$ is independent of time.

Which signify that the motion (vibration) of all the degrees of freedom are synchronised - mode shape is fixed and is written as :

$$\vec{X}=\begin{Bmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_n \end{Bmatrix}$$

Eigen values and Eigen vectors

- Substituting the assumed solution into the eqn. of motion gives:

$$[m]\vec{X}\ddot{T}(t)+[k]\vec{X}T(t)=\vec{0}$$

in scalar form:

$$\left(\sum_{j=1}^{n} m_{ij}X_j\right)\ddot{T}(t)+\left(\sum_{j=1}^{n} k_{ij}X_j\right)T(t)=0 \qquad i=1,2,3.....,n$$

which gives:

$$-\frac{\ddot{T}(t)}{T(t)}=\frac{\sum_{j=1}^{n} k_{ij}X_j}{\sum_{j=1}^{n} m_{ij}X_j} \qquad i=1,2,3.....,n$$

$$-\frac{\ddot{T}(t)}{T(t)}=\frac{\sum_{j=1}^{n} k_{ij}X_j}{\sum_{j=1}^{n} m_{ij}X_j}=\omega^2 \qquad \text{or :} \qquad \ddot{T}(t)+\omega^2T(t)=0$$

Eigen values and Eigen vectors

Then :

$$\sum_{j=1}^{n}\left(k_{ij}-\omega^2 m_{ij}\right)X_j=0 \qquad i=1,2,3......,n$$

or in matrix form:

$$\left[\,[k]-\omega^2[m]\,\right]\vec{X}=\vec{0} \qquad (a)$$

as found previously, the solution to the above can be written as :

$$T(t)=C_1\cos(\omega t+\phi)$$

- This solution reveals that the degrees of freedom can vibrate harmonically at the same frequency ω and phase angle ϕ as long as the frequency satisfies eqn. (a) which represents a set on *n* linear homogeneous equations.
- For non-trivial solutions, the determinant of the coefficient matrix must be zero which gives the *characteristic equation*:

$$\left|k_{ij}-\omega^2 m_{ij}\right|=\left|[k]-\omega^2[m]\right|=0$$

- This is known as the eigenvalue problem, where ω^2 is the eigenvalue and ω the natural frequency of the system.
- Expansion of the characteristic equation gives an n^{th} order polynomial in terms of ω^2 the solution of which produces *n* real and positive roots when the mass and stiffness matrices are symmetric and positive.
- The *n* natural frequencies are in ascending order $\omega_1 \le \omega_2 \le \omega_3 \le .... \le \omega_n$ with ω_1 being the fundamental natural frequency.

Eigen values and Eigen vectors

If we let :

$$\lambda = \frac{1}{\omega^2}$$

Equation (a) becomes:

$$[\lambda[k] - [m]]\vec{X} = \vec{0}$$

and multiplying both sides by $[k]^{-1}$ gives :

$$[\lambda[I] - [D]]\vec{X} = \vec{0}$$

or

$$\lambda[I]\vec{X} = [D]\vec{X}$$

where $[D] = [k]^{-1}[m]$ is the ***dynamical matrix***.

for a non-trivial solution the determinant of the characteristic eqn. must be zero:

$$|\lambda[I] - [D]| = 0$$

- Expanding gives an n^{th} degree polynomial in terms of λ.
- This form lends itself to obtaining solutions by numerical (computer) methods to determine the roots of a polynomial equation.