MECHANICAL VIBRATIONS

Course Name: B.Tech-ME

Semester: 7th

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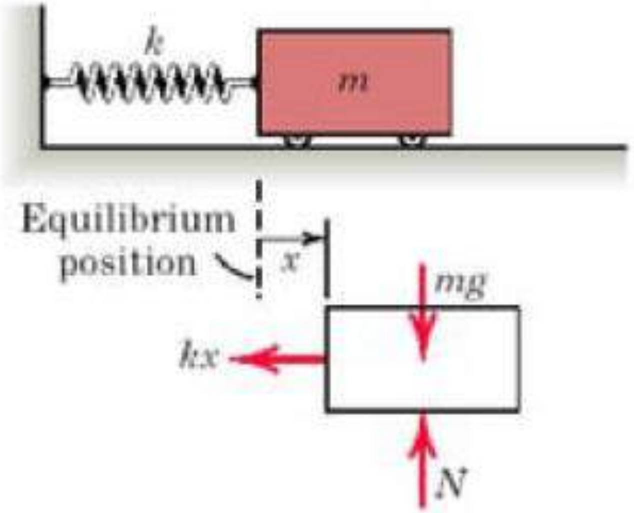
Free Undamped Vibrations Single DOF

- Recall: Free vibrations → system given initial disturbance and oscillates free of external forces.
- Undamped: no decay of vibration amplitude
- Single DoF:
  - mass treated as rigid, limped (particle)
  - Elasticity idealised by single spring
  - only one natural frequency.



- The equation of motion can be derived using
  - Newton's second law of motion
  - D'Alembert's Principle,
  - The principle of virtual displacements and,
  - The principle of conservation of energy.

UNIT: II

Free Undamped Vibrations Single DOF

Using Newton's second law of motion to develop the **equation of motion.**

1. Select suitable coordinates
2. Establish (static) equilibrium position
3. Draw free-body-diagram of mass
4. Use FBD to apply Newton's second law of motion:

   "Rate of change of momentum = applied force"

$$F(t) = \frac{d}{dt}\left(m\frac{dx(t)}{dt}\right)$$

As m is constant

$$F(t) = m\frac{d^2x(t)}{dt^2} = m\ddot{x}$$

For rotational motion

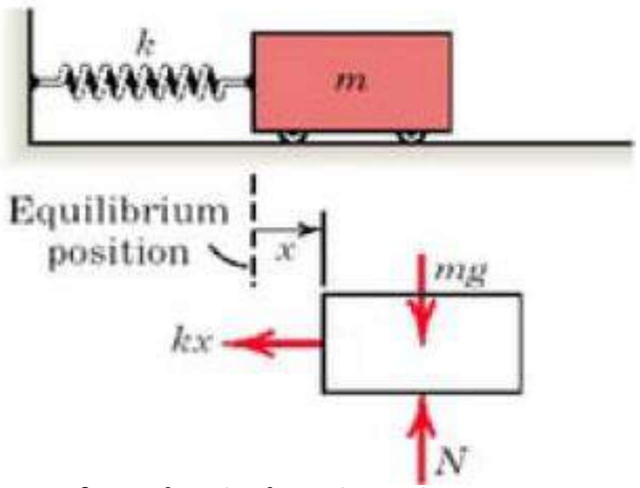
$$M(t) = J\ddot{\theta}$$

For the free, undamped single DoF system

$$F(t) = -kx = m\ddot{x}$$

or

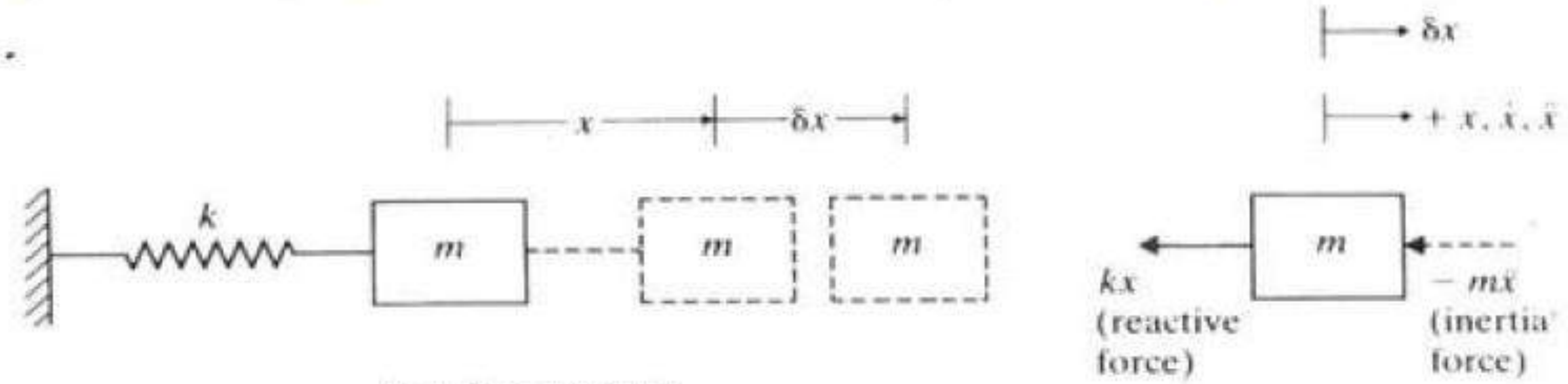
$$m\ddot{x} + kx = 0$$



Free Undamped Vibrations Single DOF

Principle of virtual displacements:

- "When a system in equilibrium under the influence of forces is given a virtual displacement. The total work done by the virtual forces = 0"
- Displacement is imaginary, infinitesimal, instantaneous and compatible with the system



(a) Mass under a displacement x

(b) Free-body diagram

- When a virtual displacement *dx* is applied, the sum of work done by the spring force and the inertia force are set to zero:

$$-(kx)\delta x \; -(m\ddot{x})\delta x = 0$$

- Since $dx \neq 0$ the equation of motion is written as:

$$kx + m\ddot{x} = 0$$

UNIT: II

Free Undamped Vibrations Single DOF

Principle of conservation of energy:

- No energy is lost due to friction or other energy-dissipating mechanisms.
- If no work is done by external forces, the system total energy = constant
- For mechanical vibratory systems:

$$KE + PE = constant$$

$$or$$

$$\frac{d}{dt}(KE + PE) = 0$$

- Since

$$KE = \frac{1}{2}m\dot{x}^2 \quad and \quad PE = \frac{1}{2}kx^2$$

$$then$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$$

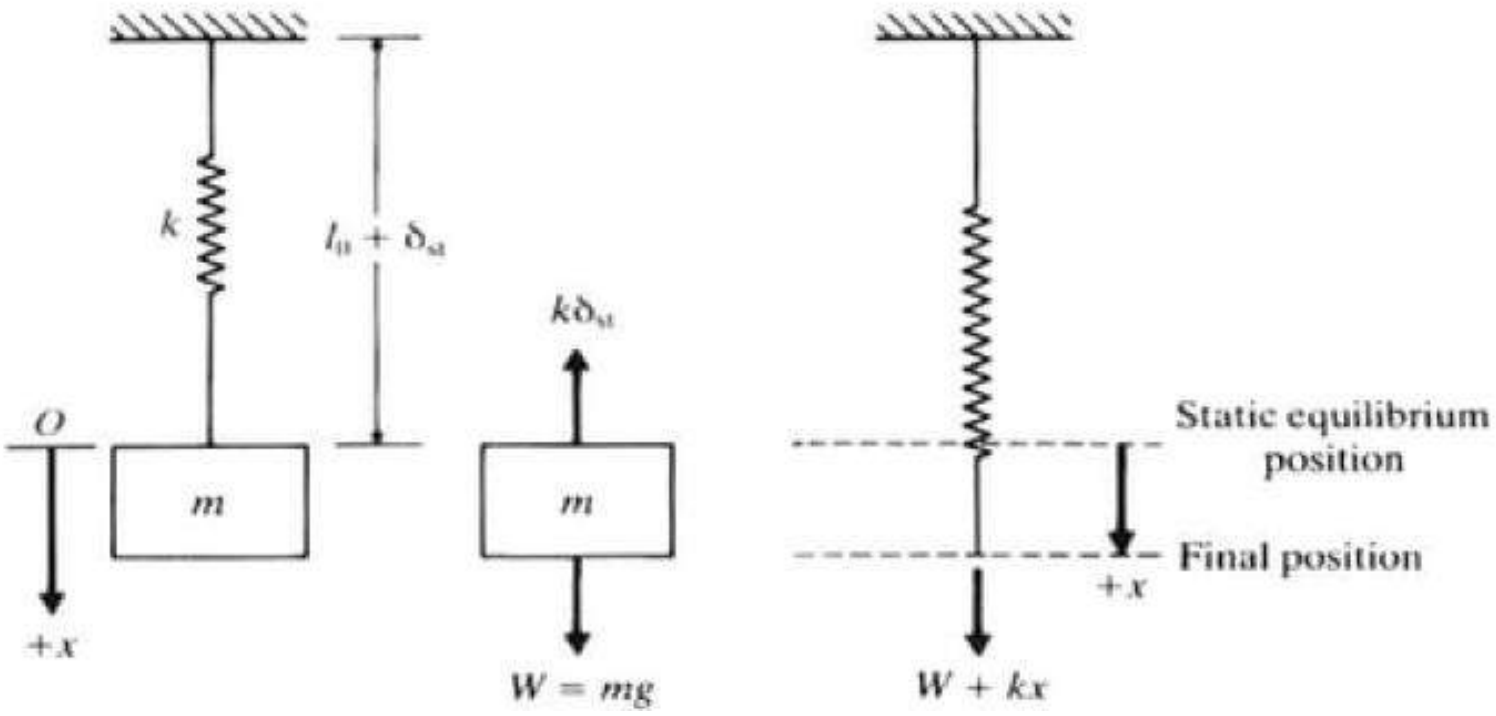
$$or$$

$$m\ddot{x} + kx = 0$$

UNIT: II

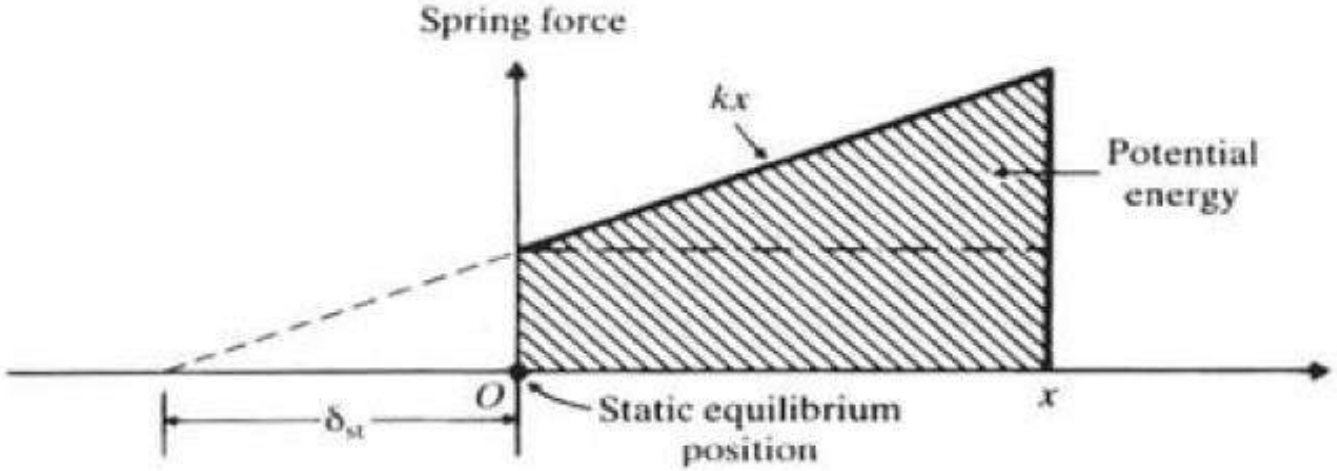
Free Undamped Vibrations Single DOF

Vertical mass-spring system:



Free Undamped Vibrations Single DOF

Vertical mass-spring system:



- From the free body diagram:, using Newton's second law of motion:

$$m\ddot{x} = -k(x + \delta_{st}) + mg$$
$$since \quad k\delta_{st} = mg$$
$$m\ddot{x} + kx = 0$$

- Note that this is the same as the eqn. of motion for the horizontal mass-spring system

∀ ∴ if x is measured from the static equilibrium position, gravity (weight) can be ignored

- This can be also derived by the other three alternative methods.

Free Undamped Vibrations Single DOF

- **The solution to the differential eqn. of motion.**
- As we anticipate oscillatory motion, we may propose a solution in the form:

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

$$or$$

$$x(t) = Ae^{i\omega_n t} + Be^{-i\omega_n t}$$

$$alternatively, if \ \ we\ let\ \ s = \pm i\omega_n$$

$$x(t) = Ce^{\pm st}$$

- By substituting for x(t) in the eqn. of motion:

$$C(ms^2 + k) = 0$$

$$since \quad c \neq 0,$$

$$ms^2 + k = 0 \qquad \neg \ Characteristic\ equation$$

$$and$$

$$s = \pm i\omega_n = \pm\sqrt{\frac{k}{m}} \qquad \neg \ roots = eigenvalues$$

$$or$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

UNIT: II

Free Undamped Vibrations Single DOF

- **The solution to the differential eqn. of motion.**
- Applying the initial conditions to the general solution: $x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$

$$x_{(t=0)} = A = x_0 \quad \textit{initial displacement}$$

$$\dot{x}_{(t=0)} = B\omega_n = \dot{x}_0 \quad \textit{initial velocity}$$

- The solution becomes:

$$x(t) = x_0\cos(\omega_n t) + \frac{\dot{x}_0}{\omega_n}\sin(\omega_n t)$$

$$\textit{if we let} \quad A_0 = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right]^{1/2} \quad \textit{and} \quad \phi = a\tan\left(\frac{x_0\omega_n}{\dot{x}_0}\right) \quad \textit{then}$$

$$x(t) = A_0\sin(\omega_n t + \phi)$$

- This describes motion of harmonic oscillator:
  - Symmetric about equilibrium position
  - Thru equilibrium: velocity is maximum & acceleration is zero
  - At peaks and valleys, velocity is zero and acceleration is maximum
  - ∀ $\omega_n = \sqrt{(k/m)}$ is the natural frequency

Free Undamped Vibrations Single DOF

- Note: for vertical systems, the natural frequency can be written as:

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$since \quad k = \frac{mg}{\delta_{st}}$$

$$\omega_n = \sqrt{\frac{g}{\delta_{st}}} \quad or \quad f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$$

- **Torsional vibration.**
- Approach same as for translational system. Laboratory exercise.

Free Undamped Vibrations Single DOF

- **Compound pendulum.**
- Given an initial angular displacement or velocity, system will oscillate due to gravitational acceleration.
- Assume rigid body → single DoF

Restoring torque:

$mgd \sin\theta$

$\therefore$ *Equation of motion :*

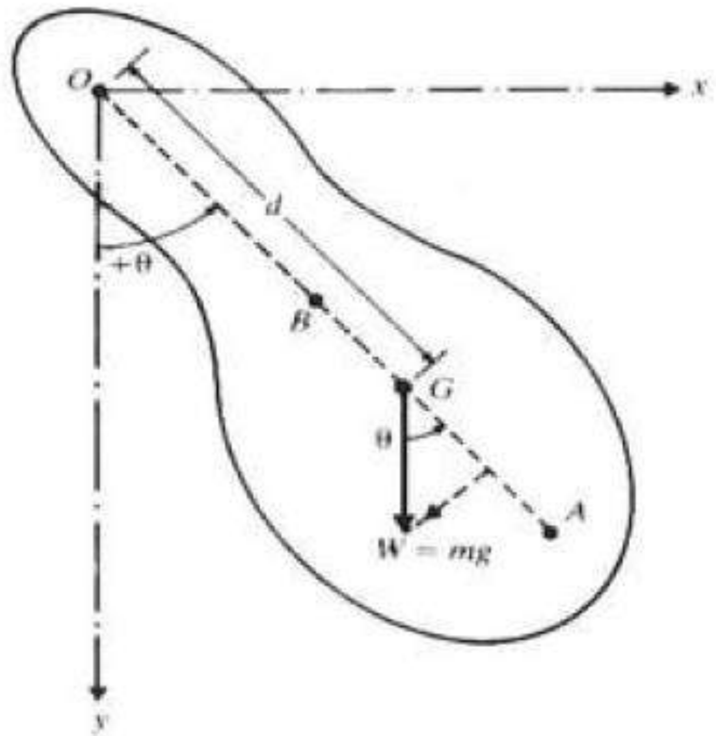
$J_o\ddot{\theta} + mgd \sin\theta = 0 \quad \neg$ *nonlinear* 2^{nd} *order ODE*

Linearity is approximated if $\sin\theta \approx \theta$ *Therefore :*

$J_o\ddot{\theta} + mgd\theta = 0$

Natural frequency :

$$\omega_n = \sqrt{\frac{mgd}{J_o}}$$



UNIT: II

Free Undamped Vibrations Single DOF

Natural frequency :

$$\omega_n = \sqrt{\frac{mgd}{J_o}}$$

since for a simple pendulum

$$\omega_n = \sqrt{\frac{g}{l}}$$

Then, $l = \frac{J_o}{md}$ *and since* $J_o = mk_o^2$ *then*

$$\omega_n = \sqrt{\frac{gd}{k_o^2}} \quad \text{and} \quad l = \frac{k_o^2}{d}$$

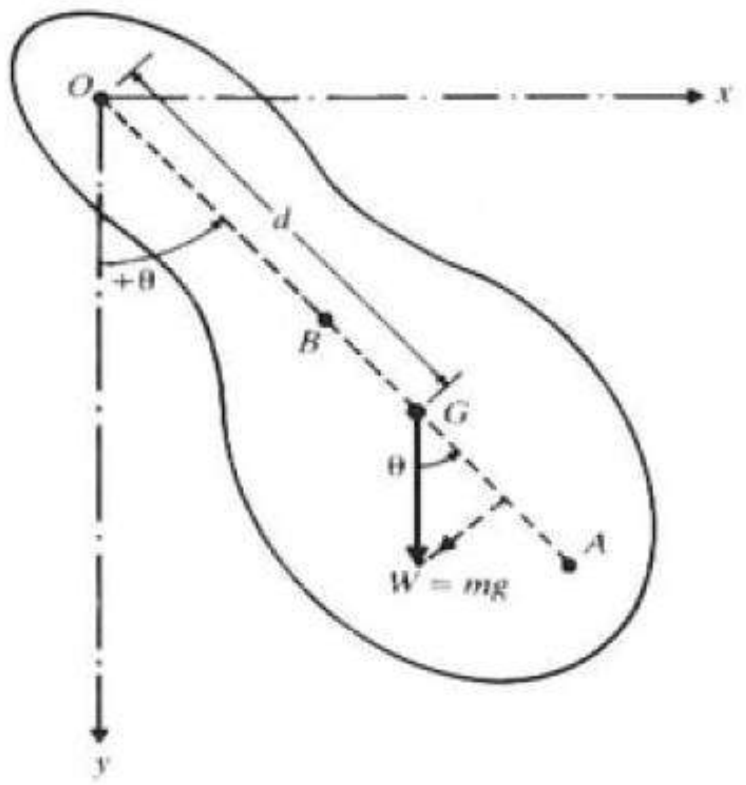
Applying the parallel axis theorem $k_o^2 = k_G^2 + d^2$

$$l = \frac{k_G^2}{d} + d$$

Let $l = GA + d = OA$

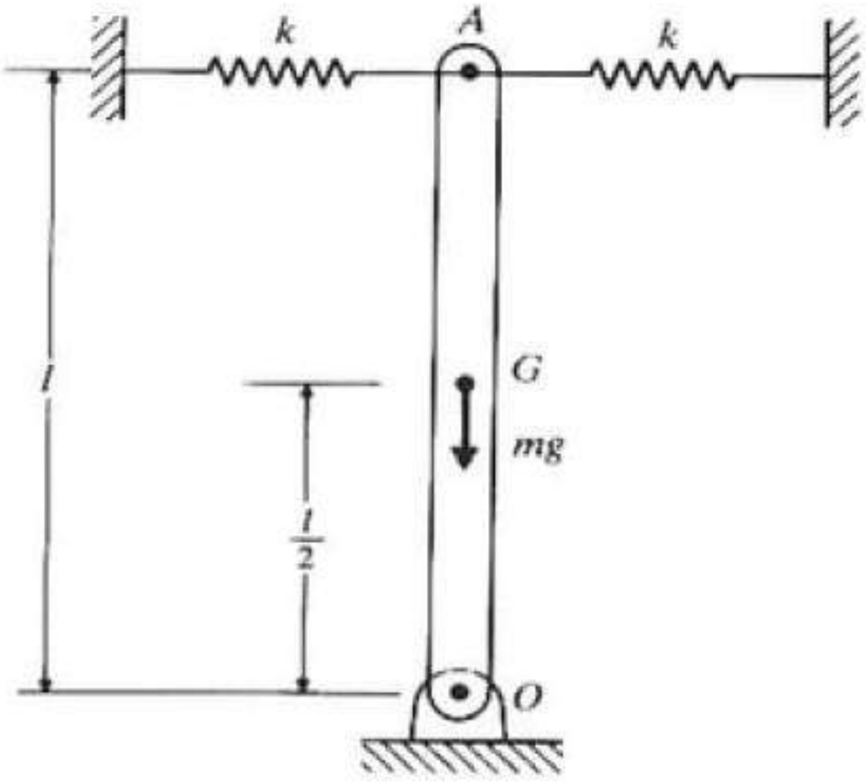
$$\omega_n = \sqrt{\frac{g}{k_o^2 / d}} = \sqrt{\frac{g}{l}} = \sqrt{\frac{g}{OA}}$$

The location A $\left(GA = \frac{k_G^2}{d}\right)$ *is the "centre of percussion"*



Free Undamped Vibrations Single DOF

- **Stability.**
- Some systems may have inherent instability



Free Undamped Vibrations Single DOF

- **Stability.**
- Some systems may have inherent instability
- When the bar is deflected by θ,

The spring force is :

$2kl\sin\theta$

The gravitational force thru G is :

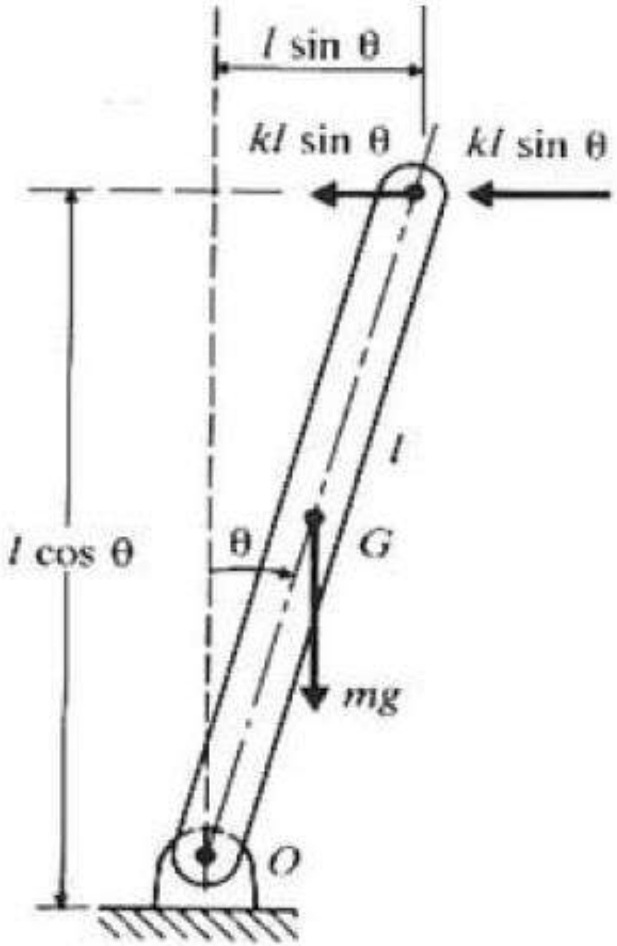
mg

The inertial moment about O due to the angular acceleration $\ddot{\theta}$ *is :*

$$J_o\ddot{\theta} = \frac{ml^2}{3}\ddot{\theta}$$

The eqn. of motion is written as :

$$\frac{ml^2}{3}\ddot{\theta} + (2kl\sin\theta)l\cos\theta - mg\frac{l}{2}\sin\theta = 0$$



Free Undamped Vibrations Single DOF

For small oscillations, $\sin\theta = \theta$ *and* $\cos\theta = 1$. *Therefore*

$$\frac{ml^2}{3}\theta + 2kl^2\theta - \frac{mgl}{2}\theta = 0$$

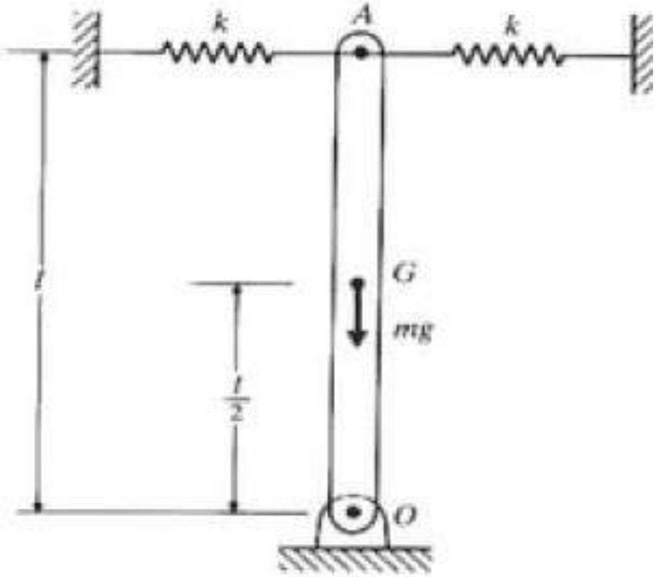
or

$$\ddot{\theta} + \left(\frac{12kl^2 - 3mgl}{2ml^2}\right)\theta = 0$$

The solution to the eqn. of motion depends of the sign of ( )

(1) If ( ) >0, the resulting motion is oscillatory (simple harmonic) with a natural frequency

$$\omega_n = \sqrt{\left(\frac{12kl^2 - 3mgl}{2ml^2}\right)}$$



Free Undamped Vibrations Single DOF

$$\ddot{\theta} + \left(\frac{12kl^2 - 3mgl}{2ml^2}\right)\theta = 0$$

(2) If ( ) =0, the eqn. of motion reduces to:

$$\ddot{\theta} = 0$$

The solution is obtained by integrating twice yielding :

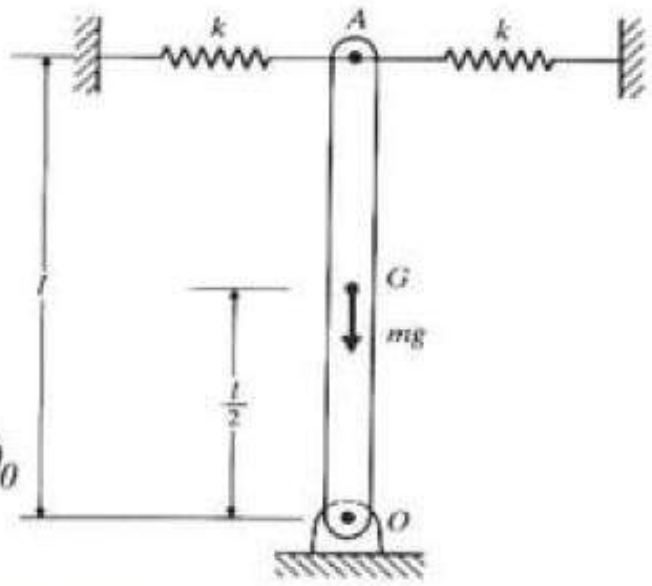
$$\theta(t) = C_1 t + C_2$$

Applying initial conditions $\theta(t=0) = \theta_0$ *and* $\dot{\theta}(t=0) = \dot{\theta}_0$

$$\theta(t) = \dot{\theta}_0 t + \theta_0$$

Which shows a linear increase of angular displ. at constant velocity.

And if $\dot{\theta}_0 = 0$ *the bar remains in static equilibrium at* $\theta(t) = \theta_0$



Free Undamped Vibrations Single DOF

$$\ddot{\theta} + \left(\frac{12kl^2 - 3mgl}{2ml^2}\right)\theta = 0$$

(3) If ( ) < 0, we define:

$$\alpha = -\left(\frac{12kl^2 - 3mgl}{2ml^2}\right) = \left(\frac{3mgl - 12kl^2}{2ml^2}\right)$$

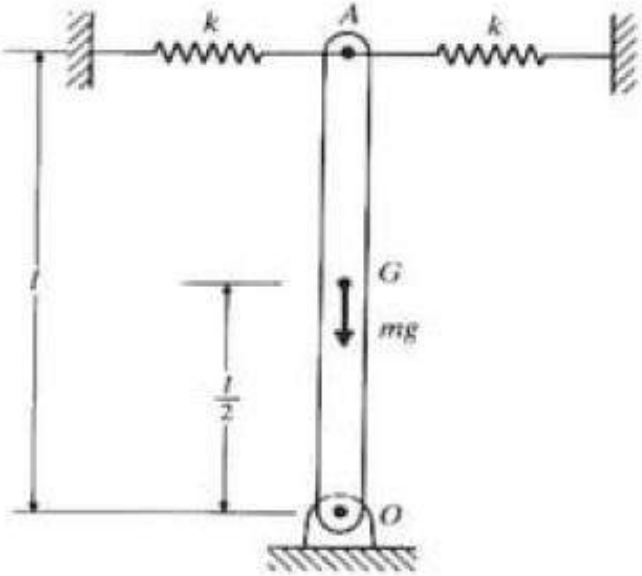
The solution of the eq. of motion is :

$$\theta(t) = B_1 e^{\alpha t} + B_2 e^{-\alpha t}$$

Applying initial conditions $\theta(t=0) = \theta_0$ *and* $\dot{\theta}(t=0) = \dot{\theta}_0$

$$\theta(t) = \frac{1}{2\alpha}\left[\left(\alpha\theta_0 + \dot{\theta}_0\right)e^{\alpha t} + \left(\alpha\theta_0 - \dot{\theta}_0\right)e^{-\alpha t}\right]$$

which shows that θ(t) increases exponentially with time and is therefore unstable because the restoring moment (springs) is less than the non-restoring moment due to gravity.



Free Undamped Vibrations Single DOF

- **Rayleigh's Energy method to determine natural frequency**
- Recall: Principle of conservation of energy:

$$T_1 + U_1 = T_2 + U_2$$

- Where T_1 and U_1 represent the energy components at the time when the kinetic energy is at its maximum ($\therefore U_1=0$) and T_2 and U_2 the energy components at the time when the potential energy is at its maximum ($\therefore T_2=0$)

$$T_1 + 0 = 0 + U_2$$

- For harmonic motion

$$T_{max} = U_{max}$$

Free Undamped Vibrations Single DOF

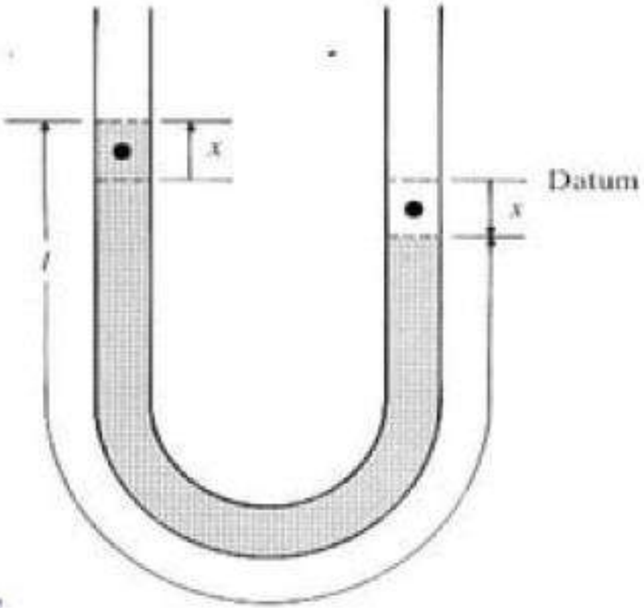
- **Rayleigh's Energy method to determine natural frequency: Application example:**
- Find minimum length of mercury u-tube manometer tube so that f_n of fluid column < 2 Hz.
- Determine U_{max} and T_{max}:
- Umax = potential energy of raised fluid column + potential energy of depressed fluid column.

$$U = mg\left.\frac{x}{2}\right|_{raised} + mg\left.\frac{x}{2}\right|_{depressed}$$

$$= (Ax\gamma)\left.\frac{x}{2}\right|_{raised} + (Ax\gamma)\left.\frac{x}{2}\right|_{depressed}$$

$$= A\gamma x^2$$

A : cross sectional area and γ : specific weight of mercury



- Kinetic energy:

$$T = \frac{1}{2}(\text{mass of mercury col})\,\text{vel}^2$$

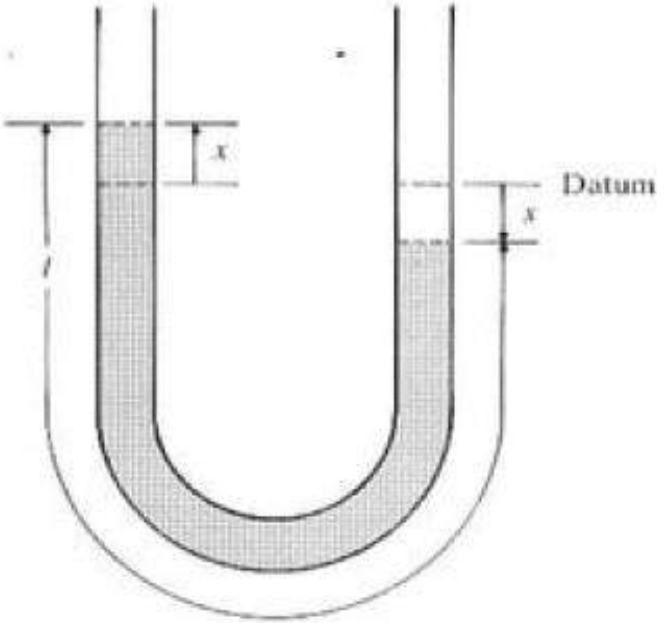
$$= \frac{1}{2}\left(\frac{Al\gamma}{g}\right)\dot{x}^2$$

Free Undamped Vibrations Single DOF

- **Rayleigh's Energy method to determine natural frequency: Application example:**
- If we assume harmonic motion:

$$x(t) = X\cos(2\pi f_n t) \quad \text{where } X \text{ is the max. displacement}$$

$$\dot{x}(t) = 2\pi f_n X \sin(2\pi f_n t) \quad \text{where } 2\pi f_n X \text{ is the max. velocity}$$



- Substituting for the maximum displacement and velocity:

$$U_{max} = A\gamma X^2 \quad \text{and} \quad T_{max} = \frac{1}{2}\left(\frac{Al\gamma}{g}\right)(2\pi f_n)^2 X^2$$

$$U_{max} = T_{max} \quad \therefore \quad A\gamma X^2 = \frac{1}{2}\left(\frac{Al\gamma}{g}\right)(2\pi f_n)^2 X^2$$

$$f_n = \frac{1}{2\pi}\sqrt{\left(\frac{2g}{l}\right)}$$

- Minimum length of column:

$$f_n = \frac{1}{2\pi}\sqrt{\left(\frac{2g}{l}\right)} \le 1.5\ Hz$$

$$l \ge 0.221\ m$$

Free Single DOF Vibration + Viscous Damping

- Recall: viscous damping force ∝ velocity:

$$F = -c\dot{x} \qquad c = damping\ constant\ or\ coefficient\ [Ns/m]$$

Applying Newton's second law of motion to obtain the eqn. of motion:

$$m\ddot{x} = -c\dot{x} - kx \qquad or \qquad m\ddot{x} + c\dot{x} + kx = 0$$

If the solution is assumed to take the form:

$$x(t) = Ce^{st} \qquad where\ s = \pm i\omega_n$$

$$then: \quad \dot{x}(t) = sCe^{st} \quad and \quad \ddot{x}(t) = s^2Ce^{st}$$

Substituting for $x, \dot{x}$ and $\ddot{x}$ in the eqn. of motion

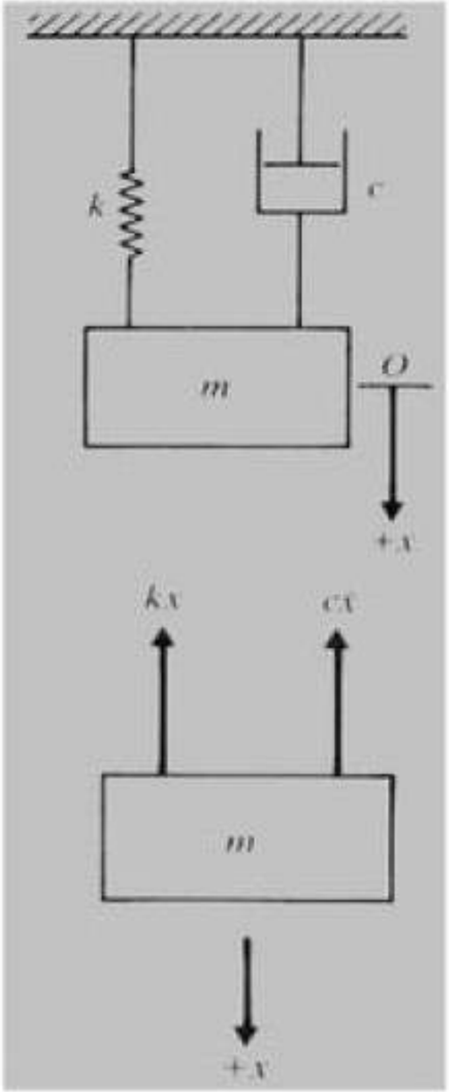
$$ms^2 + cs + k = 0$$

The root of the characteristic eqn. are:

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

The two solutions are:

$$x_1(t) = C_1e^{s_1t} \qquad and \qquad x_2(t) = C_2e^{s_2t}$$



UNIT: II

Free Single DOF Vibration + Viscous Damping

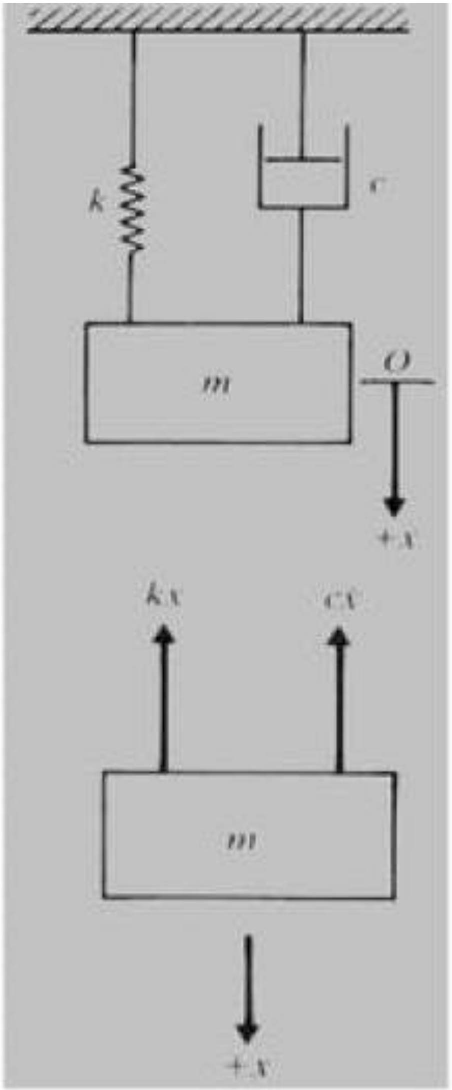
- The general solution to the Eqn. Of motion is:

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

or

$$x(t) = C_1 e^{\left\{-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}\right\}t} + C_2 e^{\left\{-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}\right\}t}$$

where C_1 and C_2 are arbitrary constants determined from the initial conditions.



UNIT: II

Free Single DOF Vibration + Viscous Damping

- **Critical damping (c_c)**: value of c for which the radical in the general solution is zero:

$$\left(\frac{c_c}{2m}\right)^2 - \left(\frac{k}{m}\right) = 0 \qquad or \qquad c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n = 2\sqrt{km}$$

- **Damping ratio (ζ)**: damping coefficient : critical damping coefficient.

$$\zeta = \frac{c}{c_c} \qquad or \qquad \frac{c}{2m} = \frac{c}{c_c}\frac{c_c}{2m} = \zeta\omega_n$$

The roots can be re-written :

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

And the solution becomes :

$$x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

- The response x(t) depends on the roots s_1 and s_2 → the behaviour of the system is dependent on the damping ratio ζ.

Free Single DOF Vibration + Viscous Damping

$$x(t) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

- When ζ <1, the system is underdamped. (ζ^2-1) is negative and the roots can be written as:

$$s_1 = \left(-\zeta + i\sqrt{1-\zeta^2}\right)\omega_n \quad and \quad s_2 = \left(-\zeta - i\sqrt{1-\zeta^2}\right)\omega_n$$

And the solution becomes :

$$x(t) = C_1 e^{\left(-\zeta+i\sqrt{1-\zeta^2}\right)\omega_n t} + C_2 e^{\left(-\zeta-i\sqrt{1-\zeta^2}\right)\omega_n t}$$

$$x(t) = e^{-\zeta\omega_n t}\left\{C_1 e^{\left(i\sqrt{1-\zeta^2}\right)\omega_n t} + C_2 e^{\left(-i\sqrt{1-\zeta^2}\right)\omega_n t}\right\}$$

$$x(t) = e^{-\zeta\omega_n t}\left\{(C_1+C_2)\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + i(C_1-C_2)\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

$$x(t) = e^{-\zeta\omega_n t}\left\{C_1'\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + C_2'\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

$$x(t) = Xe^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t+\phi\right) \quad or \quad x(t) = X_0 e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t-\phi_o\right)$$

Where C'_1, C'_2; X, ϕ and X_o, ϕ_o are arbitrary constant determined from initial conditions.

Free Single DOF Vibration + Viscous Damping

$$x(t)=e^{-\zeta\omega_n t}\left\{C_1' \cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right)+C_2' \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

- For the initial conditions:

$$x(t=0)=x_0 \quad and \quad \dot{x}(t=0)=\dot{x}_0$$

Then

$$C_1'=x_0 \quad and \quad C_2'=\frac{\dot{x}_0+\zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}$$

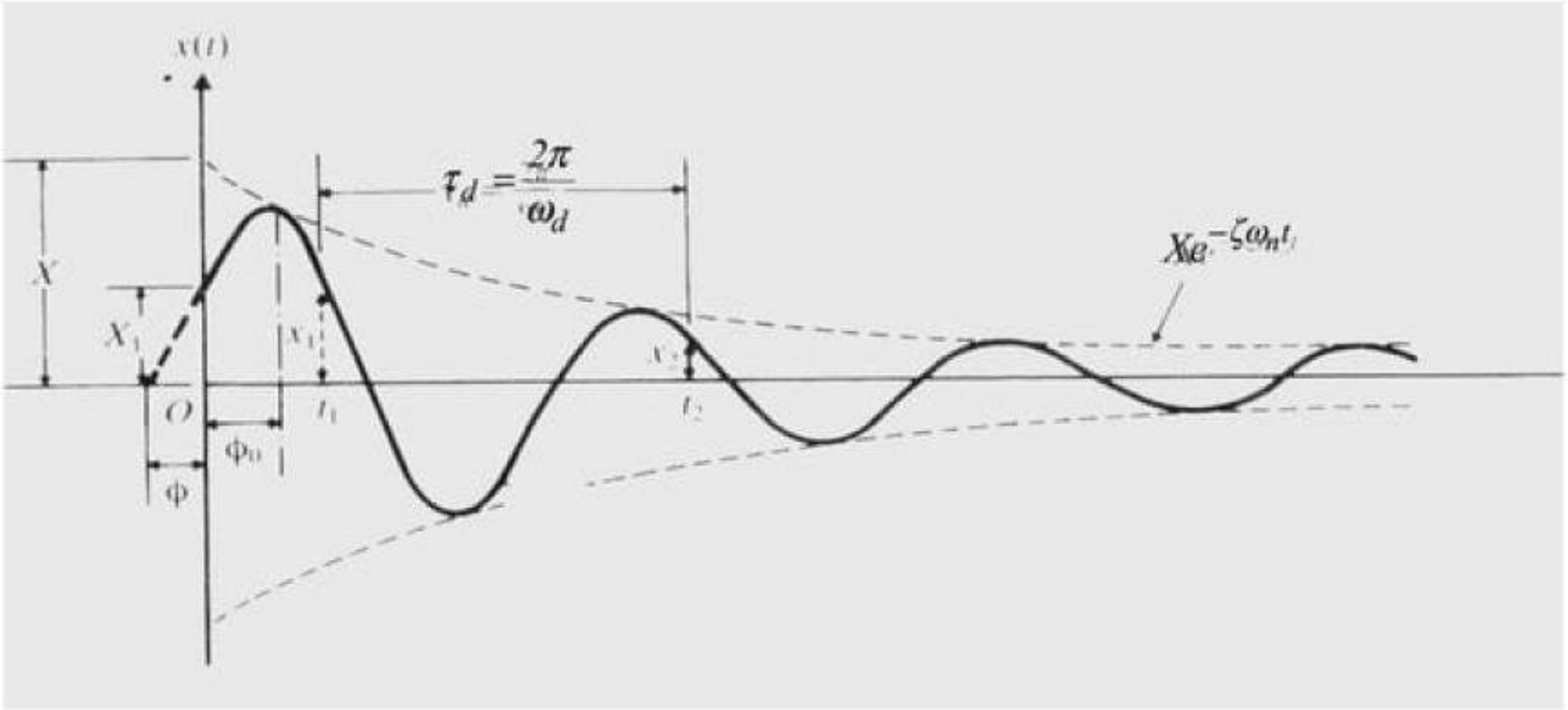
Therefore the solution becomes

$$x(t)=e^{-\zeta\omega_n t}\left\{x_0 \cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right)+\frac{\dot{x}_0+\zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

- This represents a decaying (damped) harmonic motion with angular frequency $\sqrt{(1-\zeta^2)}\omega_n$ also known as the damped natural frequency. The factor $e^{()}$ causes the exponential decay.

UNIT: II

Free Single DOF Vibration + Viscous Damping



Exponentially decaying harmonic – free SDoF vibration with viscous damping .

Underdamped oscillatory motion and has important engineering applications.

Free Single DOF Vibration + Viscous Damping

$$x(t) = Xe^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right) \quad or \quad x(t) = X_0 e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi_o\right)$$

The cons tan ts (X, ϕ) and (X_0, ϕ_0) representing the magnitude and phase become :

$$X = X_0 = \sqrt{\left(C_1'\right)^2 + \left(C_2'\right)^2}$$

$$\phi = a\tan\left(\frac{C_1'}{C_2'}\right) \quad and \quad \phi_0 = a\tan\left(-\frac{C_2'}{C_1'}\right)$$

Free Single DOF Vibration + Viscous Damping

- When $\zeta = 1$, $c = c_c$, system is critically damped and the two roots to the eqn. of motion become:

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n$$

and solution is

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$

Applying the initial conditions $x(t=0) = x_0$ *and* $\dot{x}(t=0) = \dot{x}_0$ *yields*

$$C_1 = x_0$$

$$C_2 = \dot{x}_0 + \omega_n x_0$$

The solution becomes :

$$x(t) = \left[x_0 + \left(\dot{x}_0 + \omega_n x_0 \right) t \right] e^{-\omega_n t}$$

- As $t \rightarrow \infty$, the exponential term diminished toward zero and depicts ***aperiodic*** motion

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Free Single DOF Vibration + Viscous Damping

- When $\zeta > 1$, $c>c_c$, system is overdamped and the two roots to the eqn. of motion are real and negative:

$$s_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$

$$s_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$

with $s_2 \neq s_1$ *and the initial conditions* $x(t=0) = x_0$ *and* $\dot{x}(t=0) = \dot{x}_0$

the solution becomes :

$$x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

where

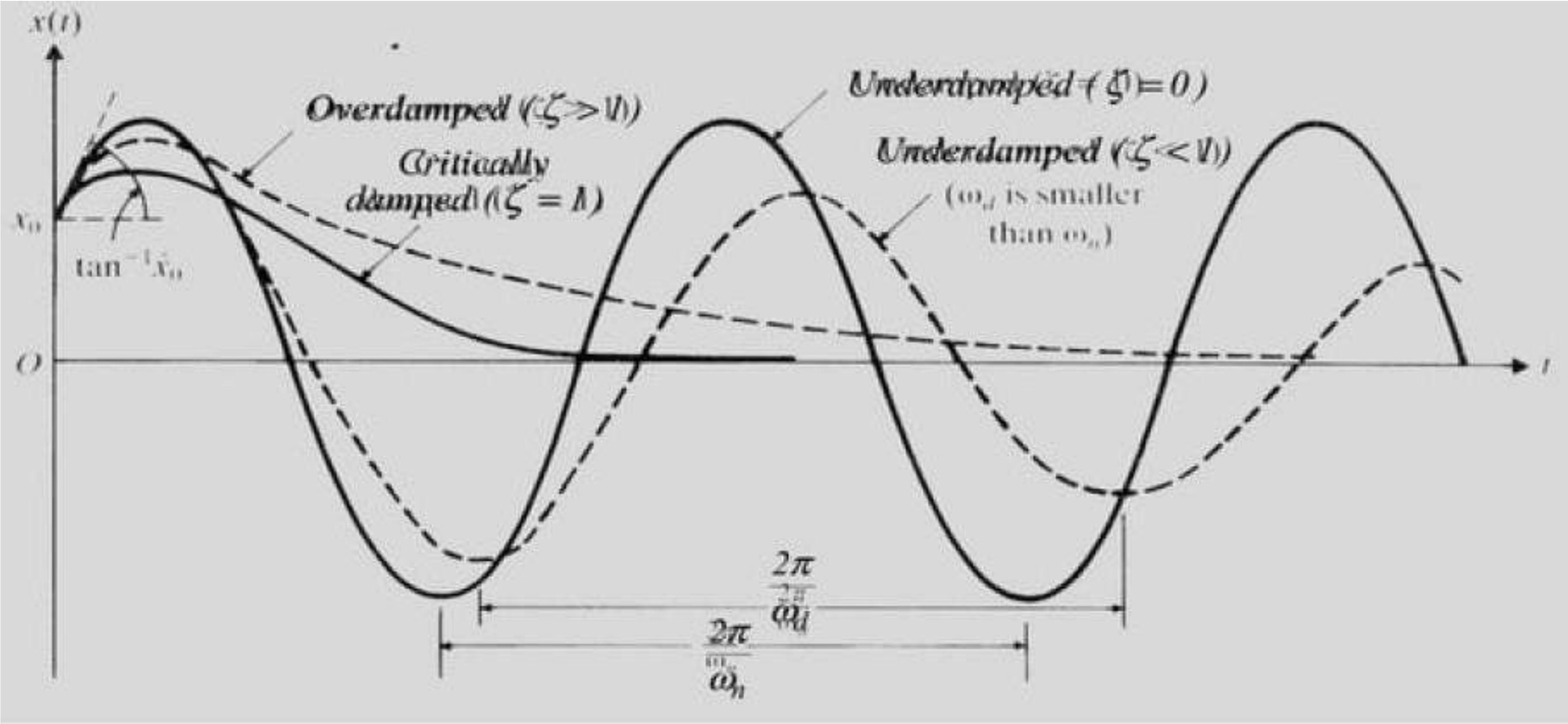
$$C_1 = \frac{x_0\omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$C_2 = \frac{-x_0\omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

Which shows ***aperiodic*** motion which diminishes exponentially with time.

UNIT: II

Free Single DOF Vibration + Viscous Damping



Critically damped systems have lowest required damping for aperiodic motion and mass returns to equilibrium position in shortest possible time.

UNIT: II

Free Single DOF Vibration + Viscous Damping

- **Logarithmic decrement:** Natural logarithm of ratio of two successive peaks (or troughs) in an exponentially decaying harmonic response.
- Represents the rate of decay
- Used to determine damping constant from experimental data.
- Using the solution for underdamped systems:

$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1} \cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta\omega_n t_2} \cos(\omega_d t_2 - \phi_0)}$$

Let $t_2 = t_1 + \tau_d = t_1 + \frac{2\pi}{\omega_d}$ *then*

$$\cos(\omega_d t_2 - \phi_0) = \cos(2\pi + \omega_d t_1 - \phi_0) = \cos(\omega_d t_1 - \phi_0)$$

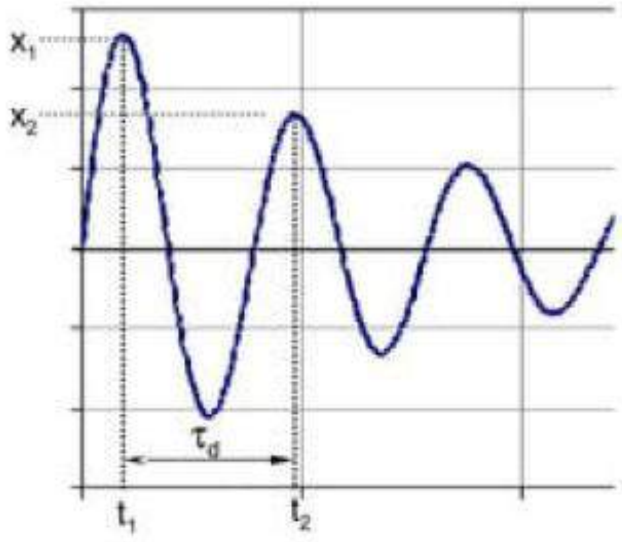
and

$$\frac{x_1}{x_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1 + \tau_d)}} = e^{\zeta\omega_n \tau_d}$$

Applying the natural ln on both sides,

the logarithmic decrement δ is obtained :

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \zeta\omega_n\tau_d = \zeta\omega_n \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi\zeta}{\omega_d}$$



UNIT: II

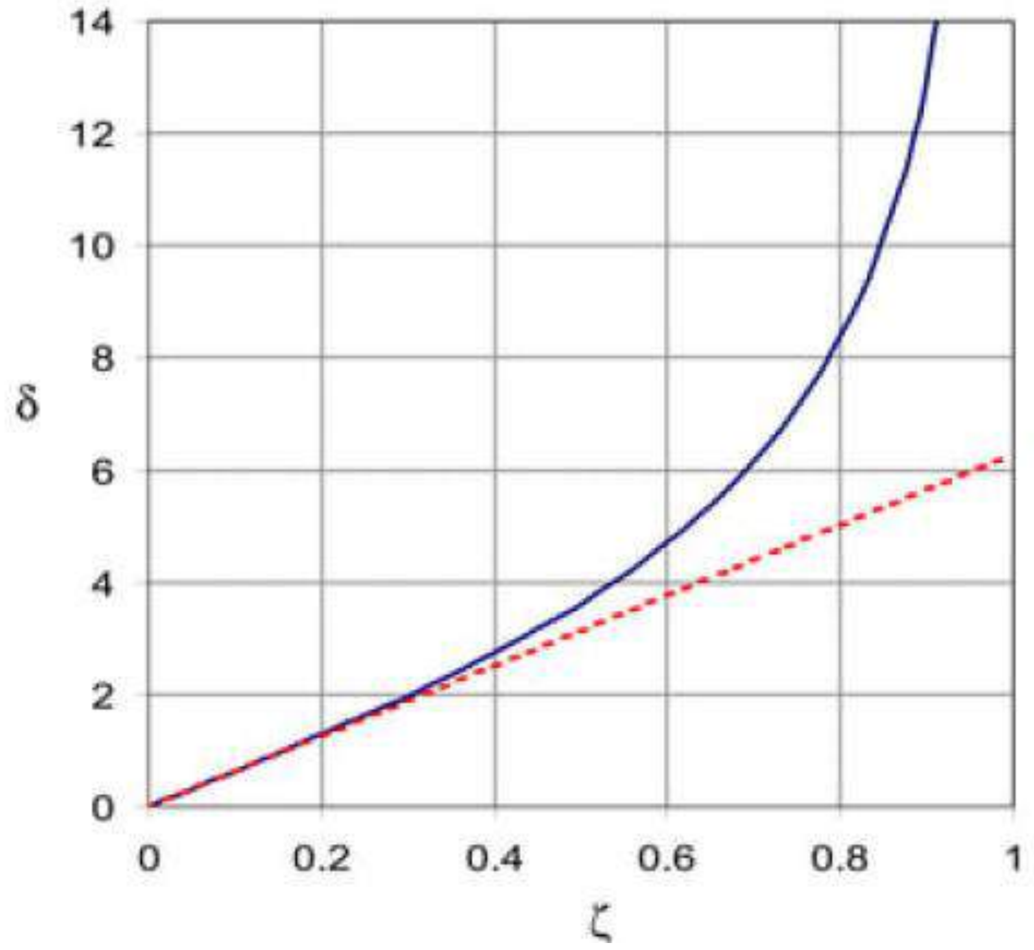
Free Single DOF Vibration + Viscous Damping

- **Logarithmic decrement:**

  For low damping $(\zeta << 1)$

  $$\delta = ln\left(\frac{x_1}{x_2}\right) = 2\pi\zeta$$

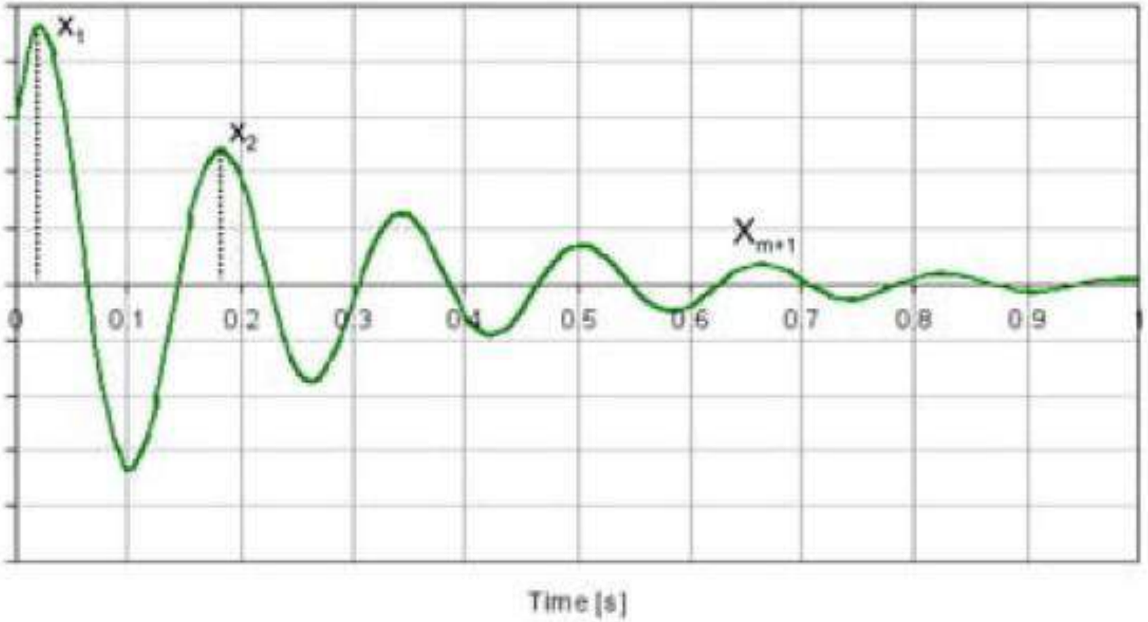
  Valid for $\zeta < .3$



UNIT: II

Free Single DOF Vibration + Viscous Damping

- **Logarithmic decrement after n cycles:**



- Since the period of oscillation is constant:

$$\frac{x_1}{x_{m+1}} = \frac{x_1}{x_2}\frac{x_2}{x_3}\frac{x_3}{x_4}\cdots\frac{x_m}{x_{m+1}}$$

Since $\frac{x_j}{x_{j+1}} = e^{\zeta\omega_n\tau_d}$ *then*

$$\frac{x_1}{x_{m+1}} = \left(e^{\zeta\omega_n\tau_d}\right)^m = e^{m\zeta\omega_n\tau_d}$$

The log arithmic decrement can therefore be obtained from a number m of successive decaying oscillations

$$\delta = \frac{1}{m}\ln\left(\frac{x_1}{x_{m+1}}\right)$$

Free Single DOF Vibration + Coulomb Damping

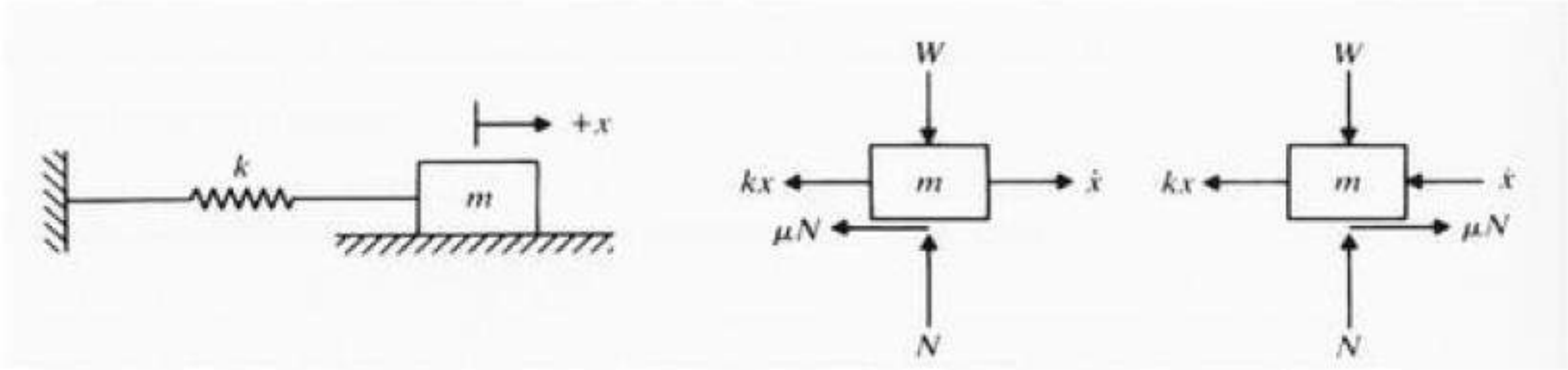
- Coulomb or dry friction dampers are simple and convenient
- Occurs when components slide / rub
- Force proportional to normal force:

$$F = \mu N$$

$$F = \mu mg \qquad \textit{for free-standing systems}$$

where μ *is the coefficient of friction.*

- Force acts in opposite direction to velocity and is independent of displacement and velocity.
- Consider SDOF system with dry friction:



Case 1.   Case 2.

Free Single DOF Vibration + Coulomb Damping

- Case 1: Mass moves from left to right. x = positive and x' is positive or x = negative and x' is positive.
- The eqn. of motion is:

$$m\ddot{x} = -kx - \mu N \quad or \quad m\ddot{x} + kx = -\mu N \quad \neg\ 2^{nd}\ order\ homogeneous\ DE$$

For which the general solution is :

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) - \frac{\mu N}{k} \qquad (1)$$

where the frequency of vibration ω_n *is* $\sqrt{\frac{k}{m}}$ *and* A_1 *and* A_2 *are constants dependent on the initial conditions of this portion of the cycle.*

- Case 2: Mass moves from right to left. x = positive and x' is negative or x = negative and x' is negative.
- The eqn. of motion is:

$$m\ddot{x} = -kx + \mu N \quad or \quad m\ddot{x} + kx = \mu N$$

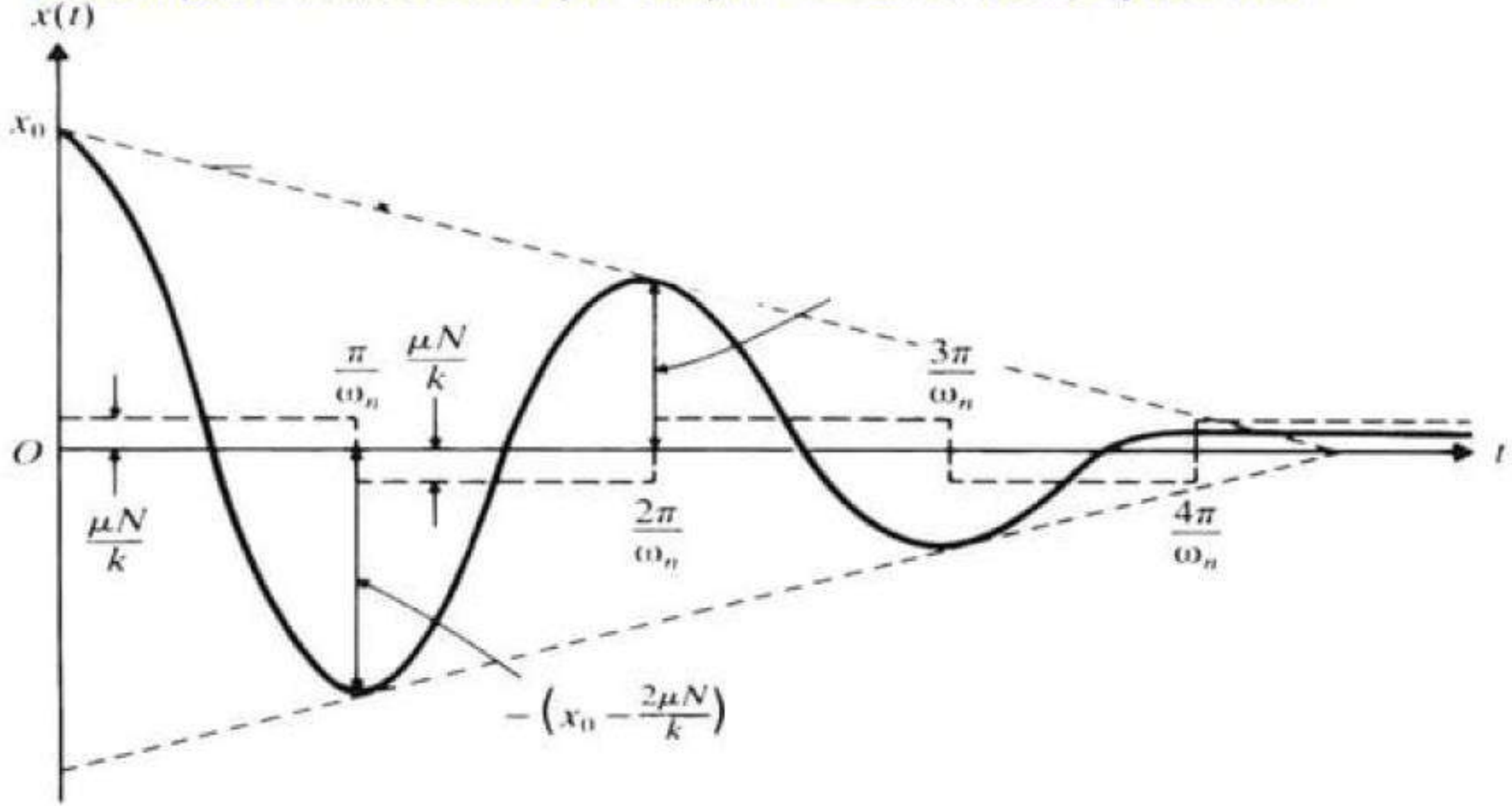
For which the general solution is :

$$x(t) = A_3 \cos(\omega_n t) + A_4 \sin(\omega_n t) + \frac{\mu N}{k} \qquad (2)$$

where the frequency of vibration ω_n *is again* $\sqrt{\frac{k}{m}}$ *and* A_3 *and* A_4 *are constants dependent on the initial conditions of this portion of the cycle.*

Free Single DOF Vibration + Coulomb Damping

- The term μN/k [m] is a constant representing the virtual displacement of the spring k under force μN. The equilibrium position oscillates between +μN/k and -μN/k 1 for each harmonic half cycle of motion.



Free Single DOF Vibration + Coulomb Damping

- To find a more specific solution to the eqn. of motion we apply the simple initial conditions:

$$x(t=0)=x_0 \quad and \quad \dot{x}(t=0)=\dot{x}_0$$

The motion starts from the extreme right (ie. velocity is zero)

Substituting into

$$x(t)=A_3\cos(\omega_n t)+A_4\sin(\omega_n t)+\frac{\mu N}{k} \qquad (2)$$

and

$$\dot{x}(t)=-A_3\omega_n\sin(\omega_n t)+A_4\omega_n\cos(\omega_n t)+0$$

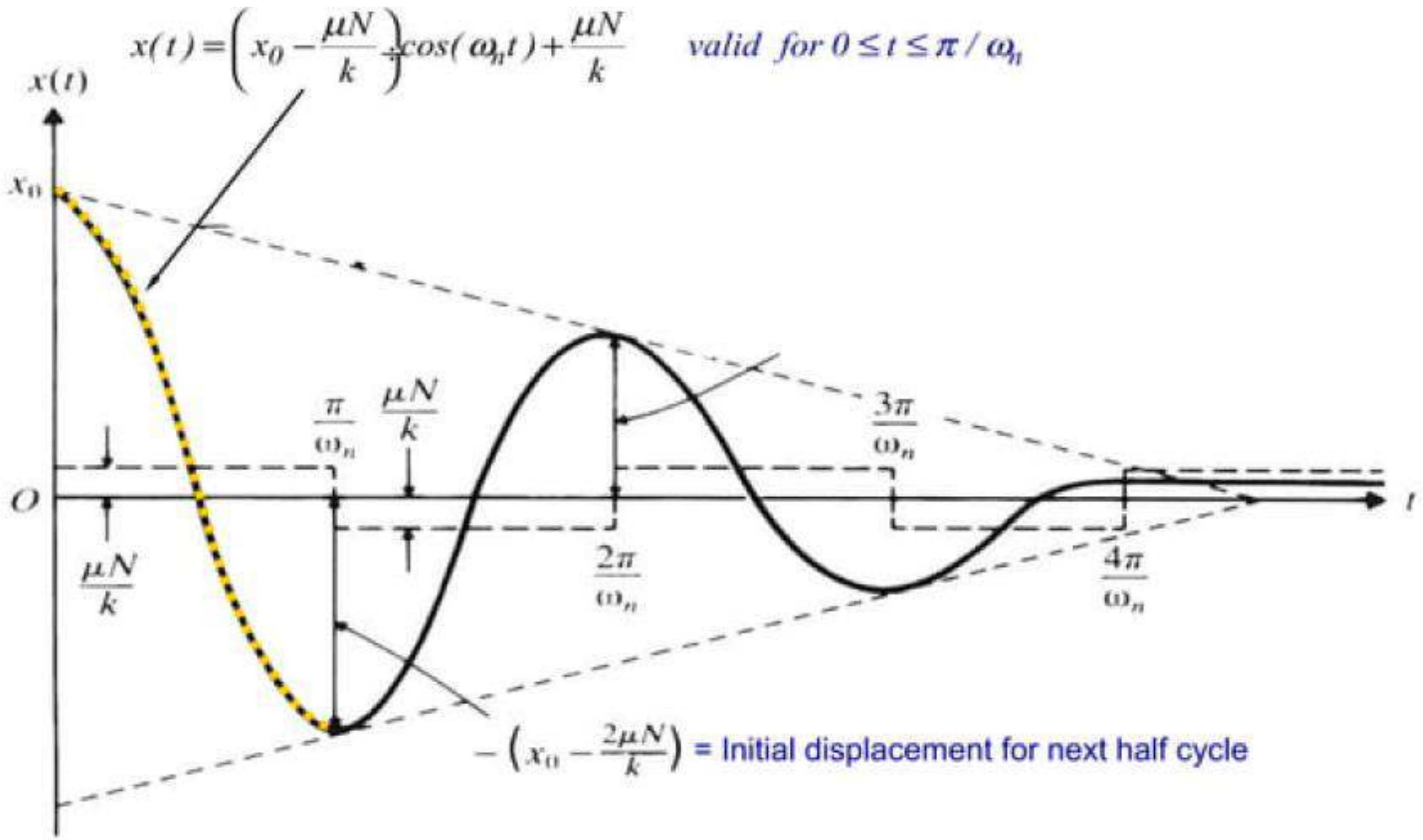
gives

$$A_3=x_0-\frac{\mu N}{k} \quad and \quad A_4=0$$

Eqn.(2) becomes

$$x(t)=\left(x_0-\frac{\mu N}{k}\right)\cos(\omega_n t)+\frac{\mu N}{k} \qquad (2a) \qquad valid\ for\ 0\le t\le \pi/\omega_n$$

Free Single DOF Vibration + Coulomb Damping



Free Single DOF Vibration + Coulomb Damping

- The displacement at π/ω_n becomes the initial displacement for the next half cycle, x_1.

$$-x_1 = x\left(t=\frac{\pi}{\omega_n}\right) = \left(x_0 - \frac{\mu N}{k}\right)\cos(\pi) + \frac{\mu N}{k} = -\left(x_0 - \frac{2\mu N}{k}\right)$$

and the initial velocity $\dot{x}(t=0)$ *is* $= \dot{x}\left(t=\frac{\pi}{\omega_n}\right)$ *in eqn (2a)*

Substituting these initial conditions int o eqn.(1)

$$x(t) = A_1\cos(\omega_n t) + A_2\sin(\omega_n t) - \frac{\mu N}{k} \qquad (1)$$

and its derivative

$$\dot{x}(t) = -\omega_n A_1\sin(\omega_n t) + \omega_n A_2\cos(\omega_n t)$$

gives

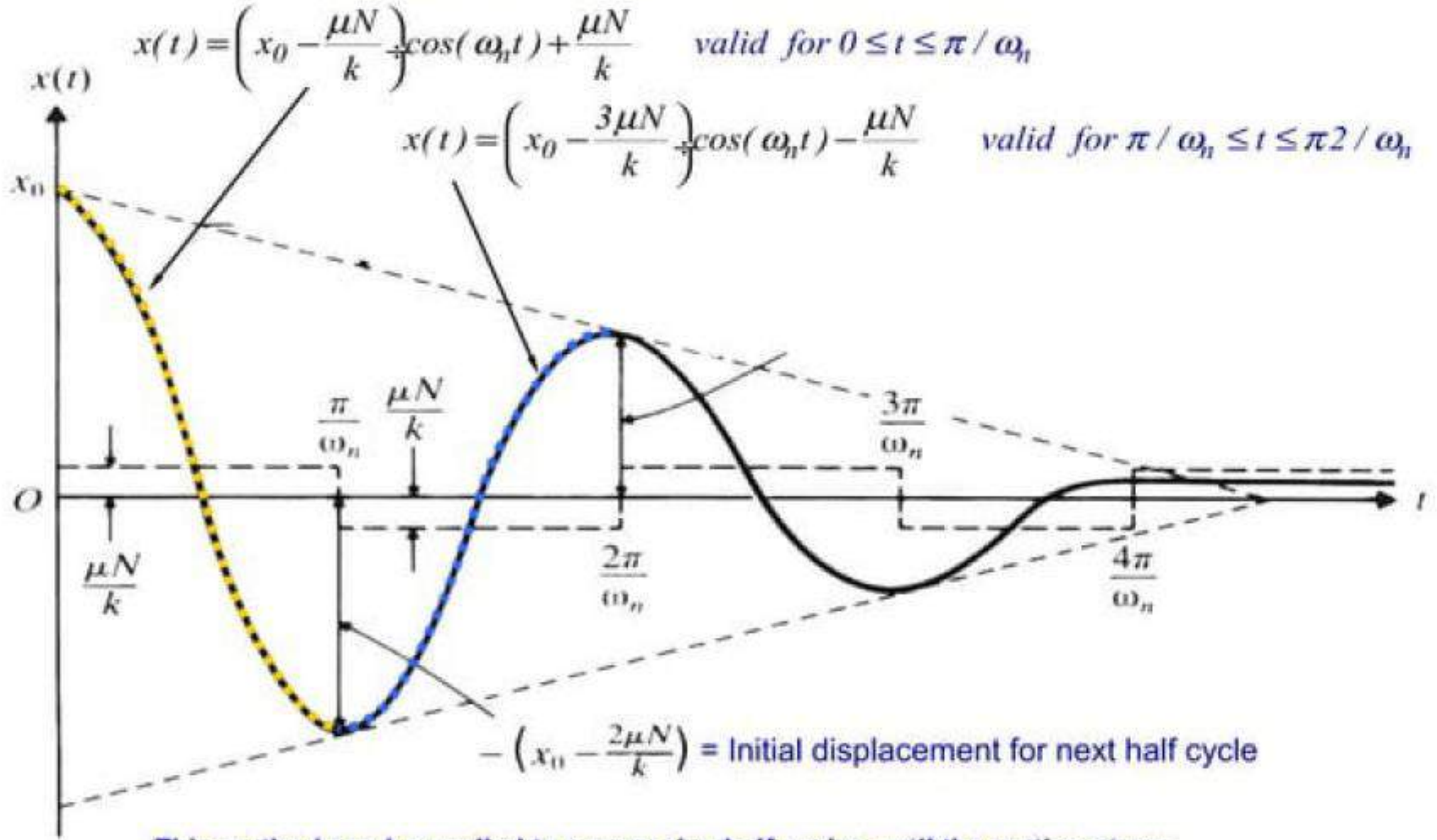
$$A_1 = x_0 - \frac{3\mu N}{k} \qquad \text{and} \qquad A_2 = 0$$

such that eqn.(1) becomes :

$$x(t) = \left(x_0 - \frac{3\mu N}{k}\right)\cos(\omega_n t) - \frac{\mu N}{k} \qquad (1a) \quad \text{valid for } \pi/\omega_n \le t \le \pi 2/\omega_n$$

UNIT: II

Free Single DOF Vibration + Coulomb Damping



This method can be applied to successive half cycles until the motion stops.

Free Single DOF Vibration + Coulomb Damping

Important features of Coulomb damping:

1. The equation of motion is nonlinear (cf. linear for viscous damping)
2. Coulomb damping **does not** alter the system's natural frequency (cf. damped natural frequency for viscous damping).
3. The motion is always periodic (cf. overdamped for viscous systems)
4. Amplitude reduces linearly (cf. exponential decay for viscous systems)
5. System eventually comes to rest – number of vibration cycles finite (cf. sustained vibration with viscous damping)
6. The final position is the permanent displacement (not equilibrium) equivalent to the friction force (cf. approaches zero for viscous systems)